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Second Structure Theorem for strong homology ☆

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Abstract

In this paper we generalize the Mardešić–Prasolov Structure Theorem for strong homology $\overline{H}_m(X;G)$ of compact Hausdorff spaces X to the theorem which connects strong homology $\overline{H}_m(\varprojlim_\lambda X_\lambda;G)$ of the inverse limit of compact Hausdorff spaces X_λ with strong homology $\overline{H}_m(X_\lambda;G)$, Čech homology $\check{H}_m(X_\lambda;G)$ and Čech cohomology $\check{H}^m(X_\lambda)$.

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1. Introduction

In [14] Mardešić and Prasolov proved the following Structure Theorem for strong homology $\overline{H}_m(X;G)$ of compact Hausdorff spaces, which we agree to call the First Structure Theorem.

Theorem 1. *For every compact Hausdorff space X, Abelian group G and integer $m \in \mathbb{Z}$, the strong homology group $\overline{H}_m(X;G)$ has a natural filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 = \overline{H}_m(X;G)$ having the following properties*:

$$F_1 \approx Pext\big(\check{H}^{m+1}(X), G\big) \approx \varprojlim{}^1 H_{m+1}(\underline{X};G)$$
$$\approx \varprojlim{}^1 Hom\big(H^{m+1}(\underline{X}), G\big); \tag{1}$$
$$F_2 \approx Ext\big(\check{H}^{m+1}(X), G\big); \tag{2}$$

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$$F_2/F_1 \approx \varprojlim Ext\big(\check{H}^{m+1}(\underline{X}), G\big); \tag{3}$$

$$F_3/F_1 \approx \check{H}_m(X; G); \tag{4}$$

$$F_3/F_2 \approx Hom\big(\check{H}^m(X), G\big) \approx \varprojlim Hom\big(H^m(\underline{X}), G\big). \tag{5}$$

The composition

$$Pext\big(\check{H}^{m+1}(X), G\big) \xrightarrow{\approx} F_1 \hookrightarrow F_2 \xrightarrow{\approx} Ext\big(\check{H}^{m+1}(X), G\big), \tag{6}$$

formed by the isomorphisms from (1) *and* (2) *and by the inclusion* $F_1 \subseteq F_2$, *coincides with the natural inclusion*

$$Pext\big(\check{H}^{m+1}(X), G\big) \hookrightarrow Ext\big(\check{H}^{m+1}(X), G\big),$$

where $p : X \to \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *is any compact polyhedra or compact ANR-resolution of* X.

Theorem 1 includes the Milnor theorem and the universal coefficient formula. In [14] the authors consider the following large commutative diagram with exact columns and rows, which contains much more information on the various groups related to the strong homology groups of compact Hausdorff spaces.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Pext(\check{H}^{m+1}(X), G) \arrow[r, "\approx"] \arrow[d] & \varprojlim^1_\lambda H_{m+1}(X_\lambda; G) \arrow[rr] \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & Ext(\check{H}^{m+1}(X), G) \arrow[r] \arrow[d] & \overline{H}_m(X; G) \arrow[r] \arrow[d] & Hom(\check{H}^m(X), G) \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & \varprojlim_\lambda Ext(H^{m+1}(X_\lambda), G) \arrow[r] \arrow[d] & \check{H}_m(X; G) \arrow[r] \arrow[d] & \varprojlim_\lambda Hom(H^m(X_\lambda), G) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{7}$$

The proof of this theorem is based on the following fact: cohomology groups $H^m(X_\lambda) = H^m(X_\lambda; \mathbb{Z})$ of a compact polyhedron or compact ANR-space X_λ with integer coefficients $\mathbb{Z}$ are finitely generated. As a consequence, the following $\varprojlim^p$ vanishing theorem is true.

Theorem 2. *For every compact Hausdorff space* X, *Abelian group* G *and integer* $m \geqslant 0$

$$\varprojlim^s_\lambda H_m(\underline{X}; G) = 0, \tag{8}$$

for all $s \geqslant 2$.

This theorem is the key to the proof of the exactness of the second column in (7).

We shall now generalize this theorem by considering a similar, but more complicated situation.

Let $X = \varprojlim_\lambda (X_\lambda, p_{\lambda\lambda'} \Lambda)$ be the inverse limit of an inverse system $\underline{X} = (X_\lambda, p_{\lambda,\lambda'}, \Lambda)$ of compact Hausdorff spaces X_λ and continuous maps, indexed by a partially ordered directed set Λ. It is well known, by the continuity property, that Čech homology and Čech cohomology of X are defined by Čech homology and Čech cohomology of X_λ as the inverse limit and the direct limit of the corresponding inverse and direct systems of groups, i.e., $\check{H}_m(X;G) = \varprojlim_\lambda \check{H}_m(X_\lambda;G)$ and $\check{H}^m(X;G) = \varinjlim_\lambda \check{H}^m(X_\lambda;G)$. We now raise the following question:

What is the relation between strong homology groups $\overline{H}_m(X;G)$ of X and strong homology groups $\overline{H}_m(X_\lambda;G)$ of X_λ, Čech homology groups $\check{H}_m(X_\lambda;G)$ of X_λ and Čech cohomology groups $\check{H}^m(X_\lambda)$ of X_λ?

The answer to this question is given here by the following Second Structure Theorem.

2. Second Structure Theorem

Theorem 3. *Let $X = \varprojlim_\lambda(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be the inverse limit of compact Hausdorff spaces X_λ and continuous maps $p_{\lambda\lambda'}: X_{\lambda'} \to X_\lambda$, $\lambda \leqslant \lambda'$, indexed by a cofinite partially ordered directed set Λ. If $\varprojlim^s_\lambda \check{H}_m(X_\lambda;G) = 0$, for every $s \geqslant 2$ and integer $m \geqslant 0$, where $\check{H}_m(X_\lambda;G)$ is the Čech homology with coefficients in an Abelian group G, then the strong homology group $\overline{H}_m(X;G)$ has a natural filtration $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4 = \overline{H}_m(X;G)$ having the following properties*:

$$\begin{aligned} F_1 &\approx \varprojlim{}^1_\lambda \overline{H}_{m+1}(X_\lambda;G) \approx \varprojlim{}^1_\lambda \check{H}_{m+1}(X_\lambda;G) \\ &\approx \varprojlim{}^1_\lambda Hom\big(\check{H}^{m+1}(X_\lambda), G\big); \end{aligned} \tag{9}$$

$$\begin{aligned} F_2 &\approx \varprojlim{}^1_{(\lambda,\mu)} H_{m+1}(Y_{(\lambda,\mu)};G) \approx Pext\big(\check{H}^{m+1}(X), G\big) \\ &\approx \varprojlim{}^1_{(\lambda,\mu)} Hom\big(\check{H}^{m+1}(Y_{(\lambda,\mu)}), G\big); \end{aligned} \tag{10}$$

$$F_3 \approx Ext\big(\check{H}^{m+1}(X), G\big); \tag{11}$$

$$F_4/F_1 \approx \varprojlim_\lambda \overline{H}_m(X_\lambda;G); \tag{12}$$

$$F_4/F_2 \approx \check{H}_m(X;G); \tag{13}$$

$$F_4/F_3 \approx Hom\big(\check{H}^m(X), G\big); \tag{14}$$

$$F_3/F_1 \approx \varprojlim_\lambda Ext\big(\check{H}^{m+1}(X_\lambda), G\big); \tag{15}$$

$$F_3/F_2 \approx \varprojlim_{(\lambda,\mu)} Ext\big(H^{m+1}(Y_{(\lambda,\mu)}), G\big); \tag{16}$$

$$F_2/F_1 \approx \varprojlim_\lambda \varprojlim{}^1_\mu H_{m+1}(Y_{(\lambda,\mu)};G) \approx \varprojlim_\lambda Pext\big(\check{H}^{m+1}(X_\lambda), G\big), \tag{17}$$

where $\underline{Y} = (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda',\mu')}, \Lambda \times M)$ is a homotopy inverse system of compact polyhedra $Y_{(\lambda,\mu)}$, indexed by a product $\Lambda \times M$ of cofinite partially ordered directed sets Λ and M such that, for every $\lambda \in \Lambda$, $\underline{q}: X_\lambda \to (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda,\mu')}, \{\lambda\} \times M)$ is an inverse limit, i.e.,

$$X_\lambda = \varprojlim_\mu \big(Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda,\mu')}\{\lambda\} \times M\big)$$

and

$$\underline{p}=(p_{(\lambda,\mu)})=\underline{q}\,p_\lambda : X \to \underline{Y}=\big(Y_{(\lambda,\mu)},\, q_{(\lambda,\mu)(\lambda',\mu')},\, \Lambda\times M\big)$$

is an ANR-expansion in the sense of Morita.

Moreover, the following three large diagrams, which contain much information on the various groups related to the strong homology groups of the compact Hausdorff space X, are commutative and their columns and rows are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \varprojlim^1_\lambda \varprojlim_\mu H_{m+1}(Y_{(\lambda,\mu)};G) \arrow[r, "\approx"] \arrow[d] & \varprojlim^1_\lambda \overline{H}_{m+1}(X_\lambda;G) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \varprojlim^1_{(\lambda,\mu)} H_{m+1}(Y_{(\lambda,\mu)};G) \arrow[r] \arrow[d] & \overline{H}_m(X;G) \arrow[r, "\beta"] \arrow[d, "\gamma"] & \check{H}_m(X;G) \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & \varprojlim_\lambda \varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)};G) \arrow[r] \arrow[d] & \varprojlim_\lambda \overline{H}_m(X_\lambda;G) \arrow[r] \arrow[d] & \varprojlim_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)};G) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{18}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \varprojlim^1_\lambda \mathit{Hom}(\check{H}^{m+1}(X_\lambda),G) \arrow[r, "\approx"] \arrow[d] & \varprojlim^1_\lambda \overline{H}_{m+1}(X_\lambda;G) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathit{Ext}(\check{H}^{m+1}(X),G) \arrow[r] \arrow[d] & \overline{H}_m(X;G) \arrow[r, "\alpha"] \arrow[d, "\gamma"] & \mathit{Hom}(\check{H}^m(X),G) \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & \varprojlim_\lambda \mathit{Ext}(\check{H}^{m+1}(X_\lambda),G) \arrow[r] \arrow[d] & \varprojlim_\lambda \overline{H}_m(X_\lambda;G) \arrow[r] \arrow[d] & \varprojlim_\lambda \mathit{Hom}(\check{H}^m(X_\lambda),G) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{19}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathit{Pext}(\check{H}^{m+1}(X),G) \arrow[r, "\approx"] \arrow[d] & \varprojlim^1_{(\lambda,\mu)}(H_{m+1}(Y_{(\lambda,\mu)});G) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathit{Ext}(\check{H}^{m+1}(X),G) \arrow[r] \arrow[d] & \overline{H}_m(X;G) \arrow[r, "\alpha"] \arrow[d, "\beta"] & \mathit{Hom}(\check{H}^m(X),G) \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & \varprojlim_{(\lambda,\mu)} \mathit{Ext}(H^{m+1}(Y_{(\lambda,\mu)}),G) \arrow[r] \arrow[d] & \check{H}_m(X;G) \arrow[r] \arrow[d] & \varprojlim_{(\lambda,\mu)} \mathit{Hom}(H^m(Y_{(\lambda,\mu)}),G) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{20}$$

Notice that the latter diagram is nothing else but the Mardešić–Prasolov diagram (7) and we have added it only for completeness. We also add the following commutative diagram with exact columns and rows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \varprojlim_\lambda Pext(\check{H}^{m+1}(X_\lambda), G) \arrow[r, "\approx"] \arrow[d] & \varprojlim_\lambda \varprojlim^1_\mu (H_{m+1}(Y_{(\lambda,\mu)}); G) \arrow[rr] \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & \varprojlim_\lambda Ext(\check{H}^{m+1}(X_\lambda), G) \arrow[r] \arrow[d] & \varprojlim_\lambda \overline{H}_m(X_\lambda; G) \arrow[rr] \arrow[d] & & Hom(\check{H}^m(X), G) \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & \varprojlim_\lambda \varprojlim_\mu Ext(H^{m+1}(Y_{(\lambda,\mu)}), G) \arrow[r] \arrow[d] & \check{H}_m(X; G) \arrow[rr] \arrow[d] & & \varprojlim_\lambda \varprojlim_\mu Hom(H^m(Y_{(\lambda,\mu)}), G) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0
\end{tikzcd} \tag{21}$$

Corollary 1. *The Second Structure Theorem is valid if the index set Λ is countable.*

Corollary 2. *The Second Structure Theorem is valid if for every $\lambda \in \Lambda$ and every integer $m \geqslant 0$ Čech homology $\check{H}_m(X_\lambda; G)$ of the compact Hausdorff spaces X_λ is finitely generated, e.g., if it is trivial.*

Corollary 3. *The Second Structure Theorem is valid if for every $\lambda \in \Lambda$ and every integer $m \geqslant 0$ the strong homology and Čech homology of a compact Hausdorff space X_λ coincide, i.e., $\overline{H}_m(X_\lambda; G) \approx \check{H}_m(X_\lambda; G)$.*

Corollary 4. *The Second Structure Theorem is valid if for every $\lambda \in \Lambda$ and every integer $m \geqslant 0$ Čech cohomology $\check{H}^m(X_\lambda)$ of the compact Hausdorff spaces X_λ is finitely generated, e.g., if it is trivial.*

Corollary 5. *The Second Structure Theorem is valid if for every $\lambda \in \Lambda$ and every integer $m \geqslant 0$ the strong homology $\overline{H}_m(X_\lambda; G)$ of the compact Hausdorff spaces X_λ coincides with $Hom(\check{H}^m(X_\lambda), G)$.*

Remark 1. By a *homotopy* inverse system $\underline{X} = (X_\alpha, p_{\alpha\alpha'}, A)$, indexed by a partially ordered directed set A, we mean a collection of spaces X_α, for each $\alpha \in A$, and continuous mappings $p_{\alpha\alpha'}: X_{\alpha'} \to X_\alpha$, for each $\alpha \leqslant \alpha'$, such that for any $\alpha \leqslant \alpha' \leqslant \alpha''$ in A composition of projections are homotopic, i.e.,

$$p_{\alpha\alpha'} p_{\alpha'\alpha''} \simeq p_{\alpha\alpha''}. \tag{22}$$

We require that $p_{\alpha\alpha} = 1_{X_\alpha}$. A morphism $\underline{f} = (f_\alpha, \phi): \underline{X} \to \underline{Y} = (Y_\beta, q_{\beta\beta'}, B)$ of homotopy inverse systems consists of a function $\phi: B \to A$ and of continuous maps $f_\beta: X_{\phi(\alpha)} \to Y_\beta$, one for each $\beta \in B$, such that whenever $\beta \leqslant \beta'$, there is an $\alpha \geqslant \phi(\beta)$, $\phi(\beta')$, for which

$$f_\beta p_{\phi(\beta)\alpha} \simeq q_{\beta\beta'} f_{\beta'} p_{\phi(\beta')\alpha}. \tag{23}$$

We say that two such morphisms $\underline{f}$ and $\underline{f}'$ are *homotopic* provided each $\beta \in B$ admits an $\alpha \in A$, $\alpha \geqslant \phi(\beta), \phi'(\beta)$, such that

$$f_\beta p_{\phi(\beta)\alpha} \simeq f'_\beta p_{\phi'(\beta)\alpha} \tag{24}$$

and denote it by $\underline{f} \simeq \underline{f}'$. Notice that if in (22)–(24) instead of $\simeq$ we put $=$, then we obtain the definitions of an inverse system, morphism of inverse systems and equivalent morphisms of inverse systems, respectively. We call them here commutative inverse systems, commutative morphisms and equivalent commutative morphisms. They are nothing else but objects from the category *inv-Top*, morphisms from *inv-Top* and morphisms from *pro-Top*.

Remark 2. An ANR-expansion in the sense of Morita [21] is a morphism $\underline{p} = (p_\alpha): X \to \underline{X}$ from a rudimentary homotopy inverse system X to a homotopy inverse system $\underline{X}$ where the following two conditions of Morita are fulfilled.

(M1) For every ANR (for metric spaces) P and every continuous map $f : X \to P$ there exist an $\alpha \in A$ and a continuous map $h : X_\alpha \to P$ such that

$$hp_\alpha \simeq f. \tag{25}$$

(M1) For every $\alpha \in A$, ANR-space P and continuous maps $f_0, f_1 : X_\alpha \to P$ such that $f_0 p_\alpha \simeq f_1 p_\alpha$, there exists an $\alpha' \geqslant \alpha$ such that

$$f_0 p_{\alpha\alpha'} \simeq f_1 p_{\alpha\alpha'}. \tag{26}$$

It is well known and easy to prove that (M1) and (M2) imply the following results:

Proposition 1. *For an arbitrary morphism $\underline{f} = (f_\alpha): X \to \underline{X} = (X_\alpha, q_{\alpha\alpha'}, A)$ from the rudimentary homotopy inverse system X to the homotopy inverse system $\underline{Y} = (Y_\beta, q_{\beta\beta'}, B)$ with ANR-spaces $Y_\beta, \beta \in B$, there exists a morphism $\underline{g} = (g_\alpha, \phi): \underline{X} \to \underline{Y}$ such that $\underline{g}\underline{p} \simeq \underline{f}$.*

Remark 3. If the index set B is cofinite, i.e., for each $\beta \in B$ the set of all predecessors of β is a finite set, then ϕ in the above proposition can be considered an increasing functions, i.e., whenever $\beta \leqslant \beta'$, then $\phi(\beta) \leqslant \phi(\beta')$ [16, p. 9]. Moreover, in this case α in (23) can be taken as $\phi(\beta')$.

For a while call $\underline{g}$ in Proposition 1 *a homotopy expansion* $\underline{f}$.

Proposition 2. *If $\underline{f} \simeq \underline{f}'$, then their homotopy expansions $\underline{g}$ and $\underline{g}'$ are also homotopic, i.e., $\underline{g} \simeq \underline{g}'$.*

To prove the main theorem we need some results.

3. Continuity Theorem

Theorem 4. *Let $\underline{p} = (p_\lambda) : X \to \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be the inverse limit of an inverse system $\underline{X}$ of compact Hausdorff spaces X_λ and continuous maps $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, $\lambda \leqslant \lambda'$, indexed by a cofinite partially ordered directed set Λ. Then there exists a homotopy inverse system $\underline{Y} = (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda',\mu')}, \Lambda \times M)$ of compact polyhedra or compact ANR-spaces $Y_{(\lambda,\mu)}$, indexed by the product $\Lambda \times M$ of cofinite directed sets Λ and M, such that, for every $\lambda \in \Lambda$, $\underline{q}_\lambda = (q_{(\lambda,\mu)}) : X_\lambda \to \underline{Y}_\lambda = (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda,\mu')}, \{\lambda\} \times M)$ is an inverse limit, i.e., $X_\lambda = \varprojlim_\mu (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda,\mu')}, \{\lambda\} \times M)$ and $\underline{p} = (p_{(\lambda,\mu)}) = \underline{q}_\lambda p_\lambda : X \to \underline{Y}$ is an ANR-expansion in the sense of Morita.*

Proof. For each $\lambda \in \Lambda$ we take an inverse system $\underline{Z}_\lambda = (Z_{\nu_\lambda}, r_{\nu_\lambda \nu'_\lambda}, N_\lambda)$ of compact polyhedra Z_{ν_λ} and continuous maps $r_{\nu_\lambda \nu'_\lambda} : Z_{\nu'_\lambda} \to Z_{\nu_\lambda}$, $\nu_\lambda \leqslant \nu'_\lambda$, indexed by a cofinite partially ordered directed set N_λ, such that $X_\lambda = \varprojlim_{\nu_\lambda} (Z_{\nu_\lambda}, r_{\nu_\lambda \nu'_\lambda}, N_\lambda)$. One can choose such inverse systems even with PL-bonding maps (see Theorem 7 in [16, p. 61]), but we do not need it here.

For every $\lambda \leqslant \lambda'$ in Λ consider a commutative morphism $\underline{r}_\lambda p_{\lambda\lambda'} : X_{\lambda'} \to \underline{Z}_\lambda$. Since each inverse limit is an ANR-expansion in the sense of Morita, by Proposition 1 and Remark 3, we can find an expansion $\underline{g}^{\lambda\lambda'} = (g^{\lambda\lambda'}_{\nu_\lambda}, \phi_{\lambda\lambda'}) : \underline{Z}_{\lambda'} \to \underline{Z}_\lambda$ with an increasing function $\phi_{\lambda\lambda'} : N_\lambda \to N_{\lambda'}$ such that for every $\nu_\lambda \leqslant \nu'_\lambda$ in N_λ we have

$$g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\phi_{\lambda\lambda'}(\nu'_\lambda)} \simeq r_{\nu_\lambda \nu'_\lambda} g^{\lambda\lambda'}_{\nu'_\lambda} \tag{27}$$

and for every $\nu_\lambda \in N_\lambda$ we have

$$g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)} \simeq r_{\nu_\lambda} p_{\lambda\lambda'}. \tag{28}$$

Moreover, since for every $\lambda \leqslant \lambda' \leqslant \lambda''$ in Λ one has $p_{\lambda\lambda'} p_{\lambda'\lambda''} = p_{\lambda\lambda''}$, evidently, $\underline{r}_\lambda p_{\lambda\lambda'} p_{\lambda'\lambda''} = \underline{r}_\lambda p_{\lambda\lambda''}$ and thus, by Proposition 2, the corresponding expansions $\underline{g}_{\lambda\lambda'} \underline{g}_{\lambda'\lambda''}$ and $\underline{g}_{\lambda\lambda''}$ are homotopic, i.e., for every ν_λ in N_λ there is a $\nu_{\lambda''} \geqslant \phi_{\lambda'\lambda''}\phi_{\lambda\lambda'}(\nu_\lambda), \phi_{\lambda\lambda''}(\nu_\lambda)$ such that

$$g^{\lambda\lambda''}_{\phi_{\lambda\lambda''}(\nu_\lambda)} r_{\phi_{\lambda\lambda''}(\nu_\lambda)\nu_{\lambda''}} \simeq g^{\lambda\lambda'}_{\nu_\lambda} g^{\lambda'\lambda''}_{\phi_{\lambda\lambda'}(\nu_\lambda)} r_{\phi_{\lambda'\lambda''}\phi_{\lambda\lambda'}(\nu_\lambda)\nu_{\lambda''}}. \tag{29}$$

Remark 4. We also require that $g^{\lambda\lambda'}_{\nu_\lambda} = 1_{Z_{\nu_\lambda}}$ and $\phi_{\lambda\lambda'} = 1_{N_\lambda}$ if $\lambda = \lambda'$.

Definition 1. We call an element $\mu = (\nu_\lambda) \in \prod_\lambda N_\lambda$ admissible, if

$$\phi_{\lambda\lambda'}(\nu_\lambda) \leqslant \nu_{\lambda'}, \tag{30}$$

whenever $\lambda \leqslant \lambda'$, and in addition all ν_λ in μ are homotopy equalizers, i.e., for every $\lambda \leqslant \lambda' \leqslant \lambda''$

$$g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)} g^{\lambda'\lambda''}_{\nu_{\lambda'}} r_{\phi_{\lambda'\lambda''}(\nu_{\lambda'})\nu_{\lambda''}} \simeq g^{\lambda\lambda''}_{\nu_\lambda} r_{\phi_{\lambda\lambda''}(\nu_\lambda)\nu_{\lambda''}}, \tag{31}$$

where $\nu_\lambda, \nu_{\lambda'}, \nu_{\lambda''}$ are from μ.

Now we define the set M as the set of all admissible elements $\mu = (\nu_\lambda) \in \prod_\lambda N_\lambda$.

We put $\mu = (\nu_\lambda) \leqslant \mu' = (\nu'_\lambda)$ provided $\nu_\lambda \leqslant \nu'_\lambda$ for every $\lambda \in \Lambda$.

Clearly, that such preordering $\leqslant$ on M is reflexive and transitive. We shall show that M is a directed set.

Indeed, let μ^1, μ^2 be any elements in M. We shall construct a $\mu \in M$ such that $\mu \geqslant \mu^1, \mu^2$, by induction on the number of predecessors of $\lambda \in \Lambda$ distinct from λ. If λ has no predecessors $\neq \lambda$, we can find a $\nu_\lambda \geqslant \nu^1_\lambda, \nu^2_\lambda$, because N_λ is directed. We take it as an initial element of μ and call it a trivial equalizer. If λ has one predecessor λ_1 distinct from λ, then λ_1 has no predecessors distinct from itself and we have already chosen ν_{λ_1}. We take any $\nu_\lambda \geqslant \phi_{\lambda_1\lambda}(\nu_{\lambda_1})$, ν^1_λ, ν^2_λ and call it also a trivial equalizer. If λ has two predecessors distinct from itself, then we consider two possibilities:

(a) $\lambda > \lambda_1, \lambda_2$ but the latter two indexes are not ordered. In this case we take any $\nu_\lambda \geqslant \phi_{\lambda_1\lambda}(\nu_{\lambda_1}), \phi_{\lambda_2\lambda}(\nu_{\lambda_2})$, ν^1_λ, ν^2_λ and call it also a trivial equalizer;

(b) $\lambda_1 < \lambda_2 < \lambda$, then $p_{\lambda_1\lambda_2} p_{\lambda_2\lambda} = p_{\lambda_1\lambda}$ and thus, $\underline{r}_{\lambda_1} p_{\lambda_1\lambda_2} p_{\lambda_2,\lambda} = \underline{r}_{\lambda_1} p_{\lambda_1\lambda}$.

Hence, by Proposition 2, $\underline{g}^{\lambda_1\lambda_2}\underline{g}^{\lambda_2\lambda} \simeq \underline{g}^{\lambda_1\lambda}$ and thus, for the already constructed $\nu_{\lambda_1}, \nu_{\lambda_2}$, one can find a $\nu'_\lambda \geqslant \phi_{\lambda_2\lambda}\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1}), \phi_{\lambda_1\lambda}(\nu_{\lambda_1})$ such that

$$g^{\lambda_1\lambda_2}_{\nu_{\lambda_1}} g^{\lambda_2\lambda}_{\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})} r_{\phi_{\lambda_2\lambda}\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})\nu'_\lambda} \simeq g^{\lambda_1\lambda}_{\nu_{\lambda_1}} r_{\phi_{\lambda_1\lambda}(\nu_{\lambda_1})\nu'_\lambda} \tag{32}$$

and we take for ν_λ any $\nu_\lambda \geqslant \nu'_\lambda, \phi_{\lambda_2\lambda}(\nu_{\lambda_2})$, ν^1_λ, ν^2_λ. It is not difficult to show that such ν_λ is a nontrivial equalizer of $\nu_{\lambda_2}, \nu_{\lambda_1}$. Indeed, since $\underline{g}^{\lambda_2\lambda}$ is a homotopy morphism, we obtain

$$g^{\lambda_2\lambda}_{\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})} r_{\phi_{\lambda_2\lambda}\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})\phi_{\lambda_2\lambda}(\nu_{\lambda_2})} \simeq r_{\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})} g^{\lambda_2\lambda}_{\nu_{\lambda_2}}. \tag{33}$$

Now, applying (32) and (33), we have

$$\begin{aligned}
&g^{\lambda_1\lambda}_{\nu_{\lambda_1 1}} r_{\phi_{\lambda_1\lambda}(\nu_{\lambda_1})\nu_\lambda}\\
&\quad = g^{\lambda_1\lambda}_{\nu_{\lambda_1 1}} r_{\phi_{\lambda_1\lambda}(\nu_{\lambda_1})\nu'_\lambda} r_{\nu'_\lambda\nu_\lambda} \simeq g^{\lambda_1\lambda_2}_{\nu_{\lambda_1}} g^{\lambda_2\lambda}_{\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})} r_{\phi_{\lambda_2\lambda}\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})\nu'_\lambda} r_{\nu'_\lambda\nu_\lambda}\\
&\quad = g^{\lambda_1\lambda_2}_{\nu_{\lambda_1}} g^{\lambda_2\lambda}_{\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})} r_{\phi_{\lambda_2\lambda}\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})\phi_{\lambda_2\lambda}(\nu_{\lambda_2})} r_{\phi_{\lambda_2\lambda}(\nu_{\lambda_2})\nu_\lambda}\\
&\quad \simeq g^{\lambda_1\lambda_2}_{\nu_{\lambda_1}} r_{\phi_{\lambda_1\lambda_2}(\nu_{\lambda_1})} g^{\lambda_2\lambda}_{\nu_{\lambda_2}} r_{\phi_{\lambda_2\lambda}(\nu_{\lambda_2})\nu_\lambda}.
\end{aligned} \tag{34}$$

Assume that ν_λ has been defined for all $\lambda \in \Lambda$ with $< k$ predecessors, distinct from λ, $k \geqslant 1$, in such a manner that all ν_λ are trivial or nontrivial equalizers $\nu_\lambda \geqslant \nu^1_\lambda, \nu^2_\lambda$. If $\lambda \in \Lambda$ has k predecessors $\lambda_1, \ldots, \lambda_k$ distinct from λ, we have already found equalizers $\nu_{\lambda_1}, \ldots, \nu_{\lambda_k}$ such that $\nu_{\lambda_i} \geqslant \nu^1_{\lambda_i}, \nu^2_{\lambda_i}$, $i = 1, \ldots, k$, and we take for ν_λ any $\nu_\lambda \geqslant \phi_{\lambda_1\lambda}(\nu_{\lambda_1}), \ldots, \phi_{\lambda_k\lambda}(\nu_{\lambda_k})$, ν^1_λ, ν^2_λ such that ν_λ is an equalizer, trivial or nontrivial, for all $\nu_{\lambda_i}, \nu_{\lambda_j}$, $i, j = 1, \ldots, k$ and $i \neq j$. This is possible because N_λ is directed and there are only a finite number of such combinations. Therefore, if $\lambda < \lambda_i < \lambda_j$, then we can repeat the construction in the above case (b). Thus, for finite numbers of equalizers, we can find ν_λ, which is great or equal to them and therefore, this ν_λ will be the needed equalizer.

By inductive construction, $\nu_\lambda \geqslant \nu^1_\lambda, \nu^2_\lambda$, for all $\lambda \in \Lambda$. Consequently, $\mu \geqslant \mu^1, \mu^2$.

Remark 5. Notice that if we take any ν_λ as initials from N_λ, λ has no predecessors distinct from itself and repeat the inductive construction of above, we shall get an admissible element μ. Thus, M is not empty. Moreover, it follows from the inductive construction that, for any fixed $\lambda \in \Lambda$, the set of all ν_λ's, which appear in an admissible μ, i.e., $\{\nu_\lambda \mid \mu = (\nu_\lambda) \in M\}$, is cofinal in N_λ. Indeed, for any fixed $\bar{\nu}_\lambda \in N_\lambda$ in the above construction of $\mu = (\nu_\lambda)$ we can choose a trivial or a nontrivial equalizer $\nu_\lambda \geqslant \bar{\nu}_\lambda$.

Now we consider the product $\Lambda \times M$ of Λ and M as a directed ordered set with $(\lambda, \mu) \leqslant (\lambda', \mu')$, provided $\lambda \leqslant \lambda'$ and $\mu \leqslant \mu'$. Let $\mu = (\nu_\lambda)$ be any element in M. We put

$$Y_{(\lambda,\mu)} = Z_{\nu_\lambda}. \tag{35}$$

If $\lambda \leqslant \lambda'$ in Λ and $\mu = (\nu_\lambda) \leqslant \mu' = (\nu'_\lambda)$ in M, then we put

$$q_{(\lambda,\mu)(\lambda',\mu')} = g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu'_{\lambda'}}. \tag{36}$$

In particular,

$$q_{(\lambda,\mu)(\lambda,\mu')} = r_{\nu_\lambda \nu'_\lambda}, \tag{37}$$

when λ is fixed, and

$$q_{(\lambda,\mu)(\lambda',\mu)} = g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu_{\lambda'}}, \tag{38}$$

when μ is fixed.

We shall show that $\underline{Y} = (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda',\mu')}, \Lambda \times M)$ is a homotopy inverse system. Indeed, if $(\lambda, \mu) \leqslant (\lambda', \mu') \leqslant (\lambda'', \mu'')$, then we have the composition

$$q_{(\lambda,\mu)(\lambda',\mu')} q_{(\lambda',\mu')(\lambda'',\mu'')} = g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu'_{\lambda'}} g^{\lambda'\lambda''}_{\nu'_{\lambda'}} r_{\phi_{\lambda'\lambda''}(\nu'_{\lambda'})\nu''_{\lambda''}}. \tag{39}$$

Since $\underline{g}^{\lambda'\lambda''}$ is a morphism of homotopy systems, one has

$$r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu'_{\lambda'}} g^{\lambda'\lambda''}_{\nu'_{\lambda'}} \simeq g^{\lambda'\lambda''}_{\phi_{\lambda\lambda'}(\nu_\lambda)} r_{\phi_{\lambda'\lambda''}(\nu'_{\lambda'})\phi_{\lambda'\lambda''}\phi_{\lambda\lambda'}(\nu_\lambda)}. \tag{40}$$

Now since $\nu_{\lambda''}$ is an equalizer of $\nu_{\lambda'}, \nu_\lambda$, one has

$$g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu_{\lambda'}} g^{\lambda'\lambda''}_{\nu_{\lambda'}} r_{\phi_{\lambda'\lambda''}(\nu_{\lambda'})} \simeq g^{\lambda\lambda''}_{\nu_\lambda} r_{\phi_{\lambda\lambda''}(\nu_\lambda)\nu_{\lambda''}}. \tag{41}$$

Consequently, by formulas (38)–(40), using the transitivity of the homotopy relation, we obtain

$$\begin{aligned}
&q_{(\lambda,\mu)(\lambda',\mu')} q_{(\lambda',\mu')(\lambda'',\mu'')} \\
&\quad = g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu'_{\lambda'}} g^{\lambda'\lambda''}_{\nu'_{\lambda'}} r_{\phi_{\lambda'\lambda''}(\nu'_{\lambda'})\nu''_{\lambda''}} \\
&\quad = g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu_{\lambda'}} r_{\nu_{\lambda'}\nu'_{\lambda'}} g^{\lambda'\lambda''}_{\nu'_{\lambda'}} r_{\phi_{\lambda'\lambda''}(\nu'_{\lambda'})\nu''_{\lambda''}} \\
&\quad \simeq g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu_{\lambda'}} g^{\lambda'\lambda''}_{\nu_{\lambda'}} r_{\phi_{\lambda'\lambda''}(\nu_{\lambda'})\phi_{\lambda'\lambda''}(\nu'_{\lambda'})} r_{\phi_{\lambda'\lambda''}(\nu'_{\lambda'})\nu''_{\lambda''}} \\
&\quad = g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu_{\lambda'}} g^{\lambda'\lambda''}_{\nu_{\lambda'}} r_{\phi_{\lambda'\lambda''}(\nu_{\lambda'})} r_{\nu_{\lambda''}\nu''_{\lambda''}} \\
&\quad \simeq g^{\lambda\lambda''}_{\nu_\lambda} r_{\phi_{\lambda\lambda''}(\nu_\lambda)\nu_{\lambda''}} r_{\nu_{\lambda''}\nu''_{\lambda''}} = g^{\lambda\lambda''}_{\nu_\lambda} r_{\phi_{\lambda\lambda''}(\nu_\lambda)\nu''_{\lambda''}} = q_{(\lambda,\mu)(\lambda'',\mu'')}.
\end{aligned} \tag{42}$$

We now define, for every fixed $\lambda \in \Lambda$, an increasing function $i:\{\lambda\} \times M \to N_\lambda$ by $i((\lambda,\mu)) = \nu_\lambda$ and we put $i_{(\lambda,\mu)} = 1_{\nu_\lambda} : Z_{\nu_\lambda} \to Z_{\nu_\lambda} = Y_{(\lambda,\mu)}$. Clearly, if $(\lambda,\mu) \leqslant (\lambda,\mu')$, then $q_{(\lambda,\mu)(\lambda,\mu')} i_{(\lambda,\mu')} = r_{\nu_\lambda \nu'_\lambda} 1_{\nu_{\lambda'}} = 1_{\nu_\lambda} r_{\nu_\lambda \nu'_\lambda} = i_{(\lambda,\mu)} r_{\nu_\lambda \nu'_\lambda}$. Thus, we have a morphism

$$\underline{i} = \big(i_{(\lambda,\mu)}\big) : \underline{Z}_\lambda \to \underline{Y}_\lambda = \big(Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda,\mu')}, \{\lambda\} \times M\big)$$

of inverse systems. Moreover, by Remark 5, the latter inverse system is a subsystem of the first one and it is cofinal in it. Moreover, $(i_{(\lambda,\mu)}, i)$ is the restriction morphism. Hence,

$$X_\lambda = \varprojlim_\mu \big(Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda,\mu')}, \{\lambda\} \times M\big).$$

Consequently,

$$\underline{q}_\lambda = \big(q_{(\lambda,\mu)}\big) = \big(i_{(\lambda,\mu)} r_{\nu_\lambda}\big) : X_\lambda \to \underline{Y}_\lambda$$

is an ANR-expansion in the sense of Morita.

Consider $\underline{p} = (p_{(\lambda,\mu)}) = \underline{q}_\lambda p_\lambda = (q_{(\lambda,\mu)} p_\lambda) : X \to \underline{Y}$. It is a homotopy mapping. Indeed, for every $(\lambda,\mu) \leqslant (\lambda',\mu')$, by definition and by formula (28), we have the following equalities and equivalences.

$$\begin{aligned} p_{(\lambda,\mu)} &= q_{(\lambda,\mu)} p_\lambda = i_{(\lambda,\mu)} r_{\nu_\lambda} p_{\lambda\lambda'} p_{\lambda'} = 1_{\nu_\lambda} r_{\nu_\lambda} p_{\lambda\lambda'} p_{\lambda'} = r_{\nu_\lambda} p_{\lambda\lambda'} p_{\lambda'} \\ &\simeq g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda(\nu_\lambda)} p_{\lambda'} = g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu'_{\lambda'}} r_{\nu'_{\lambda'}} p_{\lambda'} \\ &= g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu'_{\lambda'}} 1_{\nu'_{\lambda'}} r_{\nu'_{\lambda'}} p_{\lambda'} = g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\nu'_{\lambda'}} i_{(\lambda',\mu')} r_{\nu'_{\lambda'}} p_{\lambda'} \\ &= q_{(\lambda,\mu)(\lambda',\mu')} q_{(\lambda',\mu')} p_{\lambda'} = p_{(\lambda',\mu')} q_{(\lambda,\mu)(\lambda',\mu')}. \end{aligned} \tag{43}$$

Let $f: X \to P$ be any continuous map to an ANR-space P. Since $\underline{p} = (p_\lambda) : X \to \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an ANR-expansion, there exist a $\lambda \in \Lambda$ and a map $f_\lambda : X_\lambda \to P$ such that $f_\lambda p_\lambda \simeq f$. Since $\underline{q}_\lambda : X_\lambda \to \underline{Y}_\lambda$ is an ANR-expansion, there exist a $\mu = (\nu_\lambda) \in M$ and a map $h: Y_{(\lambda,\mu)} \to P$ such that $hq_{(\lambda,\mu)} \simeq f_\lambda$ and thus, $f \simeq hq_{(\lambda,\mu)} p_\lambda = hp_{(\lambda,\mu)}$.

Let $(\lambda,\mu) \in \Lambda \times M$ and $f_0, f_1 : Y_{(\lambda,\mu)} \to P$ be such continuous maps that $f_0 p_{(\lambda,\mu)} \simeq f_1 p_{(\lambda,\mu)}$. Then by definition, $f_0 q_{(\lambda,\mu)} p_\lambda \simeq f_1 q_{(\lambda,\mu)} p_\lambda$.

Since $\underline{p} = (p_\lambda) : X \to \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an ANR-expansion, there exists a $\lambda' \in \Lambda$ such that $f_0 q_{(\lambda,\mu)} p_{\lambda\lambda'} \simeq f_1 q_{(\lambda,\mu)} p_{\lambda\lambda'}$.

By definition and by formula (28), we have

$$f_0 g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)} \simeq f_0 q_{(\lambda,\mu)} p_{\lambda\lambda'} \simeq f_1 q_{(\lambda,\mu)} p_{\lambda\lambda'} \simeq f_1 g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)}.$$

Since $\underline{q}_{\lambda'} : X_{\lambda'} \to \underline{Z}$ is an ANR-expansion, there exists a $\bar{\nu}_{\lambda'} \in N'_\lambda$ such that

$$f_0 g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\bar{\nu}_{\lambda'}} \simeq f_1 g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\bar{\nu}_{\lambda'}}.$$

As we have already noticed there is a $\mu' = (\nu'_{\lambda'}) \in M$ such that $\nu'_{\lambda'} \geqslant \bar{\nu}_{\lambda'}$. Hence,

$$\begin{aligned} f_0 q_{(\lambda,\mu)(\lambda',\mu')} &= f_0 r_{\phi_{\lambda\lambda'}(\nu_{\lambda'})\nu'_{\lambda'}} = f_0 g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\bar{\nu}_{\lambda'}} r_{\bar{\nu}_{\lambda'}\nu'_{\lambda'}} \\ &\simeq f_1 g^{\lambda\lambda'}_{\nu_\lambda} r_{\phi_{\lambda\lambda'}(\nu_\lambda)\bar{\nu}_{\lambda'}} r_{\bar{\nu}_{\lambda'}\nu'_{\lambda'}} = f_1 q_{(\lambda,\mu)(\lambda',\mu')}. \end{aligned}$$

This concludes the proof. □

Remark 6. We have already seen that $X_\lambda = \varprojlim_\mu (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda,\mu')}, \{\lambda\} \times M)$, $\lambda \in \Lambda$, but in general, unfortunately, in such a way (we do not exclude other methods to find such an inverse system, i.e., the problem is still open) $\underline{Y}$ cannot be turned into an inverse system such that $X = \varprojlim_{(\lambda,\mu)} \underline{Y}$. Otherwise, every n-dimensional compact Hausdorff space X would be the inverse limit of compact n-dimensional polyhedra, which is not true, because there are counterexamples. Nevertheless, if $X = \bigcap_\lambda X_\lambda$ is the intersection of compact Hausdorff spaces X_λ, then we can consider them in the Tychonoff cube I^τ, for some cardinal number τ, and we can choose, for each $\lambda \in \Lambda$, an inverse system $\underline{Y} = (Y_{(\lambda,\mu)}, i_{(\lambda,\mu)(\lambda',\mu')}, \Lambda \times M)$ of compact neighborhoods $Y_{(\lambda,\mu)}$ in I^τ having a homotopy type of compact polyhedra which is cofinal in the system of all neighborhoods of X_λ with inclusions $i_{(\lambda,\mu)(\lambda',\mu')}$ as projections. Then evidently,

$$X = \bigcap_{(\lambda,\mu)} Y_{(\lambda,\mu)} = \varprojlim_{(\lambda,\mu)} \bigl(Y_{(\lambda,\mu)} i_{(\lambda,\mu)(\lambda',\mu')}, \Lambda \times M\bigr),$$

where $(\lambda,\mu) \leqslant (\lambda',\mu')$ if and only if $Y_{(\lambda',\mu')} \subseteq Y_{(\lambda,\mu)}$. This very special case should not be ignored.

Remark 7. Theorem 4 can be easily generalized to arbitrary homotopy inverse systems $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of topological spaces X_λ and continuous maps $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, $\lambda \leqslant \lambda'$, indexed by a cofinite partially ordered directed set Λ and every homotopy inverse systems $\underline{Z}_\lambda = (Z_{\mu_\lambda}, r_{\mu_\lambda \mu'_\lambda}, M_\lambda)$, $\lambda \in \Lambda$, of ANR-spaces Z_{μ_λ}, indexed by cofinite directed sets M_λ such that $\underline{p} : X \to \underline{X}$ and $\underline{r}_\lambda : X_\lambda \to \underline{Z}_\lambda$, $\lambda \in \Lambda$, are ANR-expansions in the sense of Morita. Then there exists a homotopy inverse system $\underline{Y} = (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda',\mu')}, \Lambda \times M)$ of ANR-spaces $Y_{(\lambda,\mu)}$ ($Y_{(\lambda,\mu)}$ are some Z_{μ_λ}), indexed by the product $\Lambda \times M$ of cofinite partially ordered directed sets Λ and M, such that, for each $\lambda \in \Lambda$, $\underline{q}_\lambda = (q_{(\lambda,\mu)}) : X_\lambda \to \underline{Y}_\lambda = (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda,\mu')}, \{\lambda\} \times M)$ and $\underline{p} = (p_{(\lambda,\mu)}) = \underline{q}_\lambda p_\lambda : X \to \underline{Y}$ are ANR-expansions in the sense of Morita.

4. Some properties of homotopy inverse limits

Consider the category ∂AG of cochain complexes (C,∂), i.e., sequences of Abelian groups and homomorphisms $\cdots \to C^{m-1} \xrightarrow{\partial^{m-1}} C^m \xrightarrow{\partial^m} C^{m+1} \to \cdots$, $\partial^m \partial^{m-1} = 0$, and cochain mappings, i.e., $f = (f^m) : (C,\partial) \to (C',\partial')$, where $f^m : C^m \to C'^m$ is such that $\partial'^m f^m = f^{m+1}\partial^m$. We also consider the same category as the category of chain complexes (C_m, ∂_m), and chain mappings $f = (f_m) : C_m \to C'_m$, putting $(C_m, \partial_m) = (C^{-m}, \partial^{-m})$.

Remark 8. Every graded Abelian group A^m can be consider as a cochain complex with $\partial = 0$. For a while we will call it a *simple cochain complex*. In particular, every Abelian group A has a unique cochain complex structure where $A^0 = A$, $A^i = 0$, $i \neq 0$. Here we call it shortly an *elementary cochain complex*. The trivial cochain complex (C,∂) consists of zero-groups, i.e., $C^m = 0$, for all m.

As usual, $Z^m(C) = \mathrm{Ker}(\partial^m)$ are cocycle groups, $B^m(C) = \mathrm{Im}(\partial^{m-1})$ are coboundary groups and $H^m(C) = Z^m(C)/B^m(C)$ are cohomology groups. Two canonical short exact sequences of cochain complexes

$$0 \to Z^m(C) \to C^m \to B^{m+1}(C) \to 0, \tag{44}$$

$$0 \to B^m(C) \to C^m \to C^m/B^m(C) \to 0 \tag{45}$$

imply long exact sequences of cohomologies, which split in the following short exact canonical sequences

$$0 \to B^m(C) \to Z^m(C) \to H^m(C) \to 0, \tag{46}$$

$$0 \to H^m(C) \to C^m/B^m(C) \to B^{m+1}(C) \to 0. \tag{47}$$

Definition 2. A cochain mapping $f:(C,\partial) \to (C',\partial')$ is called a weak equivalence if the induced homomorphisms $H^m(f^m): H^m(C) \to H^m(C')$ are isomorphisms for all m.

Definition 3. A graded morphism $s^m: C^m \to C'^{m-1}$ of degree -1 is called a cochain homotopy between two cochain mappings $f = (f^m)$, $g = (g^m)$, $f^m, g^m : (C^m, \partial^m) \to (C'^m, \partial'^m)$, if $\partial'^{m-1}s^m + s^{m+1}\partial^m = f^m - g^m$.

Definition 4. A cochain mapping $f:(C,\partial) \to (C',\partial')$ is called a cochain homotopy equivalence if there exists a cochain mapping $f':(C',\partial') \to (C,\partial)$ such that $f'f$ and ff' are cochain homotopic to 1_C and $1_{C'}$, respectively.

By *Ho* ∂AG we denote the homotopy category, whose objects are objects of ∂AG and the morphisms are homotopy classes $[f]$ of cochain mappings f of ∂AG.

We now consider the category *inv*-∂AG of inverse systems $\underline{C} = (C_\lambda, p_{\lambda\lambda'}, \Lambda)$, of cochain complexes $C_\lambda = (C,\partial)_\lambda$ and cochain projections $p_{\lambda\lambda'} = (p^m)_{\lambda\lambda'}$, indexed by a cofinite partially ordered directed set Λ, and the morphisms $\underline{f} = (f_\mu, \phi) = ((f^m)_\mu, \phi)$ of systems consisting of cochain mappings with increasing functions ϕ. We also consider a category *pro*-∂AG of the same objects and equivalent classes of morphisms. In the same way we can define the categories *inv-Ho* ∂AG and *pro-Ho* ∂AG.

Let Λ be a directed set. We consider for each $n \geqslant 0$ the set Λ^n of increasing sequences $\underline{\lambda} = (\lambda_0, \ldots, \lambda_n)$, $\lambda_0 \leqslant \cdots \leqslant \lambda_n$, $\lambda_i \in \Lambda$.

Definition 5. Let $\underline{C} = (C_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of cochain complexes and cochain mappings, indexed by a partially ordered directed set Λ. By homotopy inverse limit $\operatorname{ho}\varprojlim_\lambda \underline{C}$ we call the cochain complex, whose m-cochains, $m \in \mathbb{Z}$, are functions x, which assign to every $\underline{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \Lambda^n$ $(m-n)$-cochains $x_{\underline{\lambda}}$ of $C^{m-n}_{\lambda_0}$, i.e., $x_{\underline{\lambda}} \in C^{m-n}_{\lambda_0}$, $n = 0, 1, \ldots.$

The sum $x + x'$ of two m-cochains, $m \in \mathbb{Z}$, is given by $(x + x')_{\underline{\lambda}} = x_{\underline{\lambda}} + x'_{\underline{\lambda}}$. Clearly, we get a graded Abelian group.

We will now define boundary operators

$$d^m : (\operatorname{holim}_{\lambda} \underline{C})^m \to (\operatorname{holim}_{\lambda} \underline{C})^{m+1}, \quad m \in \mathbb{Z}.$$

If $n \geqslant 1$, $0 \leqslant j \leqslant n$ and $\underline{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \Lambda^n$, let $\underline{\lambda}_j \in \Lambda^{n-1}$ be obtained from $\underline{\lambda}$ by deleting the term λ_j. If $x \in (\operatorname{holim}_{\lambda} \underline{C})^m$ and $\underline{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \Lambda^n$, $n \geqslant 0$, we put

$$\left(d^m x\right)_{\underline{\lambda}} = \partial^m(x_{\underline{\lambda}}), \quad n = 0, \tag{48}$$

$$\left(d^m x\right)_{\underline{\lambda}} = (-1)^n \partial^m(x_{\underline{\lambda}}) + p_{\lambda_0\lambda_1} x_{\underline{\lambda}_0} + \sum_{j=1}^{n} (-1)^j x_{\underline{\lambda}_j}, \quad n \geqslant 1. \tag{49}$$

Clearly, d^m is a homomorphism and $d^m d^{m-1} = 0$. The proof one can find (up to a sign in formula (49)) in [9].

Remark 9. Putting $C_m = C^{-m}$ in Definition 5, we obtain the homotopy inverse limit of an inverse system of chain complexes.

Proposition 3. *The homotopy inverse limit* holim *is a covariant functor from the category inv-∂AG to the category ∂AG.*

Proof. Every morphism $\underline{f} = (f_\mu, \phi) : \underline{C} \to \underline{D} = (D_\mu, q_{\mu\mu'}, M)$ of inverse systems of cochain complexes induces a cochain mapping $\operatorname{holim} \underline{f}^m : (\operatorname{holim}_{\lambda} \underline{C})^m \to (\operatorname{holim}_{\mu} \underline{D})^m$, where

$$\left(\operatorname{holim} \underline{f}^m x\right)_{\underline{\mu}} = f^m_{\mu_0}\left(x_{\phi(\mu_0)\cdots\phi(\mu_n)}\right), \quad \underline{\mu} = (\mu_0, \ldots, \mu_n) \in M^n. \tag{50}$$

It is easy to verify (see in [9, p. 32]) that

$$d^m \operatorname{holim} \underline{f}^m = \operatorname{holim} \underline{f}^m d^m$$

and

$$\left(\operatorname{holim} \underline{g}^m \underline{f}^m\right) = \left(\operatorname{holim}_{\lambda} \underline{g}^m\right)\left(\operatorname{holim}_{\lambda} \underline{f}^m\right)$$

for the composition of $\underline{f} : \underline{C} \to \underline{D}$ and $g : \underline{D} \to \underline{T}$. □

Proposition 4. *The homotopy inverse limit* holim *is a covariant functor from the category pro-∂AG to the category Ho ∂AG.*

Proof. Let $\underline{f} : \underline{C} \to \underline{D}$ be a morphism of two inverse systems in *inv*-∂AG, given by an increasing function $\phi : M \to \Lambda$ and by cochain mappings $f_\mu : C_{\phi(\mu)} \to D_\mu$. If $\psi : M \to \Lambda$ is an increasing function, $\psi \geqslant \phi$, and $g_\mu : C_{\psi(\mu)} \to D_\mu$ is given by $g_\mu = f_\mu p_{\phi(\mu)\psi(\mu)}$, then ψ and g_μ, $\mu \in M$, also define a morphism of systems $\underline{g} : \underline{C} \to \underline{D}$. We say that $\underline{g}$ is *congruent* to $\underline{f}$. Clearly, two morphisms $\underline{f}, \underline{f}' : \underline{C} \to \underline{D}$ are equivalent, i.e., define the same morphism in *pro*-∂AG, if an only if there is a morphism $\underline{g} : \underline{C} \to \underline{D}$, which is congruent to both morphisms $\underline{f}$ and $\underline{f}'$. In [9, p. 33], it was actually shown that if $\underline{g}:\underline{C}\to\underline{D}$ is congruent to $\underline{f}:\underline{C}\to\underline{D}$, then the induced cochain mappings $\operatorname{ho\underleftarrow{lim}}\underline{f}^{\#}, \operatorname{ho\underleftarrow{lim}}\underline{g}^{\#}:(\operatorname{ho\underleftarrow{lim}}_{\lambda}\underline{C})^{\#}\to(\operatorname{ho\underleftarrow{lim}}_{\mu}\underline{D})^{\#}$ are cochain homotopic and the cochain homotopy is given by the explicit formula

$$(Hx)_{\underline{\mu}}=(-1)^{n}\sum_{k=0}^{n}(-1)^{k}f_{\mu_0}\big(x_{\phi(\mu_0)\ldots\phi(\mu_k)\psi(\mu_k)\cdots\psi(\mu_n)}\big), \tag{51}$$

where $x\in C^{m}$, $\underline{\mu}=(\mu_0,\ldots,\mu_n)\in M^{n}$, $n\geqslant 0$. We omit the details. □

Remark 10. We are working with inverse systems, indexed by a cofinite partially ordered directed set Λ, although the homotopy inverse limit is defined for arbitrary directed sets Λ. There is no loss of generality, because of the Mardešić trick [16], which to every inverse system $\underline{C}$ in *pro-∂AG*, indexed by a directed set Λ assigns an isomorphic inverse system $\underline{D}$, indexed by a directed cofinite ordered set M. Moreover, each term (bonding morphism) in $\underline{D}$ is actually a term (bonding morphism) of $\underline{C}$. And there are two morphisms $f=(f_{\mu},\phi):\underline{C}\to\underline{D}$ and $g=(g_{\lambda},\psi):\underline{D}\to\underline{C}$ such that $gf=1_{\underline{C}}$ and $fg\sim 1_{\underline{D}}$. Moreover, ϕ is an increasing function and therefore, by (50) we can define an induced cochain mapping $\operatorname{ho\underleftarrow{lim}} f^{\#}:(\operatorname{ho\underleftarrow{lim}}_{\lambda}\underline{C})^{\#}\to(\operatorname{ho\underleftarrow{lim}}_{\mu}\underline{D})^{\#}$. Unfortunately, ψ is not increasing and we cannot use (50).

5. Spectral sequences

We now need results from some spectral sequence related to $\operatorname{ho\underleftarrow{lim}}_{\lambda}\underline{C}$.

The homotopy inverse limit $\operatorname{ho\underleftarrow{lim}}_{\lambda}\underline{C}$ can be considered as a $\prod$-total bicomplex

$$(\operatorname{ho\underleftarrow{lim}}_{\lambda}\underline{C})^{m}=\prod_{s=0}^{\infty}\prod_{\underline{\lambda}\in\Lambda^{s}}C_{\lambda_0}^{m-s} \tag{52}$$

with components

$$D^{s,t}=\begin{cases}\prod_{\underline{\lambda}\in\Lambda^{s}}C_{\lambda_0}^{t}, & s\geqslant 0,\\ 0, & s<0,\end{cases} \tag{53}$$

with a boundary operator $d:K^{m}\to K^{m+1}$ on

$$K^{m}=(\mathit{Tot}\,D)^{m}=\prod_{s+t=m}D^{s,t}=\prod_{s=0}^{\infty}\prod_{\underline{\lambda}\in\Lambda^{s}}C_{\lambda_0}^{m-s}, \tag{54}$$

given by $d=\partial'+\partial''$, where $\partial':D^{s,t},\to D^{s,t+1}$, $\partial'':D^{s,t}\to D^{s+1,t}$,

$$(\partial' x)_{\underline{\lambda}}=(-1)^{s}\partial(x_{\underline{\lambda}}), \tag{55}$$

$$(\partial'' x)_{\underline{\lambda}}=p_{\lambda_0\lambda_1}x_{\underline{\lambda}_0}+\sum_{j=1}^{s}(-1)^{j}x_{\underline{\lambda}_j}. \tag{56}$$

Consider on $K = \mathit{Tot}\, D$ a natural filtration F^s (it is the well-known *second filtration* of a $\prod$-total bicomplex), which on each component K^m is given by the following formula:

$$F^s = \mathit{Ker}\left(K^m \to \prod_{k=0}^{s-1} \prod_{\underline{\lambda} \in \Lambda^k} C_{\lambda_0}^{m-k}\right). \tag{57}$$

It is clear that $\varinjlim_s F^s = K$, i.e., the filtration is exhaustive, and $\varprojlim_s F^s = 0$, i.e., the filtration is Hausdorff.

The spectral sequence related to this filtration can be given by two unraveled exact couples $(A^{s,t}, E^{s,t})$ and $(B^{s,t}, E^{s,t})$, where

$$A^{s,t} = H^{s+t} F^s, \tag{58}$$

$$B^{s,t} = H^{s+t}\big(K/F^s\big), \tag{59}$$

$$E^{s,t} = H^{s+t}\big(F^s/F^{s+1}\big). \tag{60}$$

The first one is called *injective* and the second one is called *projective*.

Moreover,

$$B_2^{s,t} = \mathrm{Im}\big(H^{s+t}\big(K/F^{s+1}\big) \xrightarrow{j_{s,s+1}^{s+t}} H^{s+t}\big(K/F^s\big)\big) \overset{\mathrm{def}}{=} \overline{H}_{(s)}^{s+t}(\underline{C}), \tag{61}$$

$$E_2^{s,t} = \varprojlim{}_\lambda^s H^t(C_\lambda). \tag{62}$$

and there are two families of exact sequences, expressed by Miminoshvili [20]:

$$\begin{aligned} 0 &\to \varprojlim{}_\lambda^1 H^{m-1}(\underline{C}) \to \overline{H}_{(2)}^m(\underline{C}) \to \overline{H}_{(1)}^m(\underline{C}) \to \varprojlim{}_\lambda^2 H^{m-1}(\underline{C}) \to \cdots \\ &\to \varprojlim{}_\lambda^s H^{m-1}(\underline{C}) \to \overline{H}_{(s+1)}^{m+s-1}(\underline{C}) \to \overline{H}_{(s)}^{m+s-1}(\underline{C}) \\ &\to \varprojlim{}_\lambda^{s+1} H^{m-1}(\underline{C}) \to \cdots, \end{aligned} \tag{63}$$

$$0 \to \varprojlim{}_s^1 \overline{H}_{(s)}^{m-1} \to H^m\big(\mathrm{ho}\varprojlim{}_\lambda(\underline{C})\big) \to \varprojlim{}_s \overline{H}_{(s)}^m(\underline{C}) \to 0. \tag{64}$$

Remark 11. For $s = 1$ the group $\overline{H}_{(1)}^m(\underline{C})$ is isomorphic to $\varprojlim_\lambda H^m(\underline{C})$, where $H^m(\underline{C}) = (H^m(C_\lambda), p_{\lambda\lambda'*}, \Lambda)$. Indeed, an arbitrary element u of $\overline{H}_{(1)}^m(\underline{C})$ is the $j_{1,2}^m$-image of an element of $H_{(2)}^m(\underline{C})$, which is given by a cocycle x of K^m/F^2. This cocycle consists of cochains $x_{\lambda_0} \in (C_{\lambda_0})^m$ and $x_{\lambda_0\lambda_1} \in (C_{\lambda_0})^{m-1}$, $\lambda_0 \leqslant \lambda_1$, such that $\partial x_{\lambda_0} = 0$, $\partial x_{\lambda_0\lambda_1} = p_{\lambda_0\lambda_1} x_{\lambda_1} - x_{\lambda_0}$. Therefore, the cohomology class $[x_{\lambda_0}] \in H^m(C_{\lambda_0})$ is defined and $p_{\lambda_0\lambda_1 *}[x_{\lambda_1}] = [x_{\lambda_0}]$, which shows that $([x_{\lambda_0}])$, $\lambda_0 \in \Lambda$, is an element of $\varprojlim_\lambda H^m(\underline{C})$. We assign this element to u. Notice that our filtration differs from the one used in [15] (we quoted this remark from there) only in a shift in the numeration, and one can find more details in [15].

Proposition 5 [4]. *If $\underline{f} : \underline{C} \to \underline{C}'$ induces isomorphisms $f^{s,t} : E_2^{s,t} \to E_2'^{s,t}$, then* $\mathrm{ho}\varprojlim \underline{f}$ *is a weak equivalence.*

Proving this proposition not only for $r = 2$ but for arbitrary $r \geqslant 2$, the authors of [4] effectively used both injective and projective couples $(A^{s,t}, E^{s,t})$ and $(B^{s,t}, E^{s,t})$. The following remark shows that Proposition 5 is true for some spectral sequences in a deeper level, i.e., in the limit form.

Remark 12. In this very case the spectral sequence is conditionally convergent in the sense of Boardman [1], i.e., $\varprojlim_s A^{s,t} = 0$ and $\varprojlim_s^1 A^{s,t} = 0$. And since it is a “right-half-plane” spectral sequence, i.e., $E^{s,t} = 0,\ s < 0$, in this case one has, for large r, $E^{s,t}_{r+1} \subseteq E^{s,t}_r$ and in this case $E^{s,t}_\infty = \varprojlim_r E^{s,t}_r$. Moreover, we can define the following very important group $RE^{s,t}_\infty = \varprojlim_r^1 E^{s,t}_r$. Boardman proved [1, p. 34], that, if for $\underline{f} : \underline{C} \to \underline{C}'$ the induced homomorphisms

$$\varprojlim_r f^{s,t}_r : E^{s,t}_\infty \to E'^{s,t}_\infty$$

and

$$\varprojlim_r^1 f^{s,t}_r : RE^{s,t}_\infty \to RE'^{s,t}_\infty$$

are isomorphisms, then $\operatorname{ho}\varprojlim \underline{f}$ is a weak equivalence. For a “whole-plane” conditionally convergent spectral sequence if in addition to the above isomorphisms, $\underline{f}$ induces an isomorphism $\underline{f}_\# : W^{s,t}_\infty \to W'^{s,t}_\infty$, then $\operatorname{ho}\varprojlim \underline{f}$ is also a weak equivalence. Here we denote by $W^{s,t}_\infty$ and $W'^{s,t}_\infty$ the corresponding obstruction groups in the spectral sequences $E^{s,t}_r$ and $E'^{s,t}_r$ that arise from the interaction of limits and colimits (more precisely, $\varprojlim_r^1$ and $\varinjlim_n$, where r and n arise from

$$\operatorname{Im}^r A^{s,t} = \operatorname{Im}\bigl(A^{s+r,t} \to A^{s,t}\bigr) \quad \text{and} \quad K_n A^{s,t} = \operatorname{Ker}\bigl(A^{s,t} \to A^{s-n,t}\bigr)$$

of the injective exact couple $(A^{s,t}, E^{s,t})$). Details see in [1, pp. 48–56].

Proposition 6. *Let $\underline{C} = (C_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of Abelian groups C_λ and homomorphisms and consider it as an inverse system of cochain complexes with the unique cochain complex structure, indexed by a partially ordered directed set Λ. Then the homotopy inverse limit $\operatorname{ho}\varprojlim_\lambda \underline{C}$ is the Roos resolution* [22]

$$R^m(\underline{C}) = \prod_{\underline{\lambda} \in \Lambda^m} C_{\lambda_0}$$

and therefore,

$$H^m\bigl(\operatorname{ho}\varprojlim_\lambda \underline{C}\bigr) = \varprojlim_\lambda^m \underline{C}.$$

Proof. By (54) and Remark 12, the only nontrivial term $C^{m-s}_{\lambda_0}$ is obtained when $m = s$, consequently $K^m = \prod_{\underline{\lambda} \in \Lambda^m} C_{\lambda_0}$ and $\partial = 0$. Thus, the boundary operator d^m coincides with the boundary operator δ^m in the Roos resolution. □

Corollary. *The derived functor $\varprojlim_\lambda^m \underline{C}$ is an invariant of isomorphisms in pro-AG.*

Theorem 5. *Let $\underline{C} = (C_{(\lambda,\mu)}, p_{(\lambda,\mu)(\lambda',\mu')}, \Lambda \times M)$ be an inverse system of cochain complexes and cochain mappings, indexed by a product $\Lambda \times M$ of cofinite partially ordered directed sets Λ and M. Then*

$$\operatorname{ho}\underleftarrow{\lim}_{(\lambda,\mu)} C_{(\lambda,\mu)} \simeq \operatorname{ho}\underleftarrow{\lim}_{\lambda} \operatorname{ho}\underleftarrow{\lim}_{\mu} C_{(\lambda,\mu)} \simeq \operatorname{ho}\underleftarrow{\lim}_{\mu} \operatorname{ho}\underleftarrow{\lim}_{\lambda} C_{(\lambda,\mu)}. \tag{65}$$

Proof. We shall prove the first cochain homotopy equivalence $\simeq$ (and thus weak equivalence) in (65). Clearly, the second equivalence is symmetric. By the definition,

$$\left(\operatorname{ho}\underleftarrow{\lim}_{(\lambda,\mu)} \underline{C}\right)^m = \prod_{n=0}^{\infty} \prod_{\underline{(\lambda,\mu)} \in (\Lambda\times M)^n} C^{m-n}_{(\lambda_0,\mu_0)}$$

is the set of all functions x, which assign to every $\underline{(\lambda,\mu)} = ((\lambda_0,\mu_0),\dots,(\lambda_n,\mu_n))$, $(\lambda_0,\mu_0) \leqslant \cdots \leqslant (\lambda_n,\mu_n)$, cochains $x_{\underline{(\lambda,\mu)}}$ in $C^{m-n}_{(\lambda_0,\mu_0)}$. On the other hand,

$$\begin{aligned}\left(\operatorname{ho}\underleftarrow{\lim}_{\lambda} \operatorname{ho}\underleftarrow{\lim}_{\mu} \underline{C}\right)^m &= \prod_{s=0}^{\infty} \prod_{\underline{\lambda}\in\Lambda^s} \underline{C}^{m-s}_{\lambda_0} = \prod_{s=0}^{\infty} \prod_{\underline{\lambda}\in\Lambda^s} \prod_{t=0}^{\infty} \prod_{\underline{\mu}\in M^t} C^{m-s-t}_{(\lambda_0,\mu_0)} \\ &= \prod_{s+t=n=0}^{\infty} \prod_{\underline{\lambda}\in\Lambda^s} \prod_{\underline{\mu}\in M^t} C^{m-s-t}_{(\lambda_0,\mu_0)}\end{aligned}$$

$$\lambda_0 \leqslant \cdots \leqslant \lambda_s \quad \text{and} \quad \mu_0 \leqslant \cdots \leqslant \mu_t,$$

are also functions y, which assign to each $\underline{\lambda} \in \Lambda^s$ first of all functions $y_{\underline{\lambda}}$ of $(\operatorname{ho}\underleftarrow{\lim}_{\mu} \underline{C}_{\lambda_0})^{m-s}$ and then to each $\underline{\mu} \in M^t$ assign cochains $((y)_{\underline{\lambda}})_{\underline{\mu}}$ in $C^{m-s-t}_{(\lambda_0,\mu_0)}$. We shall define two cochain mappings f^m from the second set of such functions to the first one and g^m from the first set of such functions to the second one. They are dual to the well-known Alexander–Whitney and Eilenberg–Zilber mappings [11]. So, we put

$$\left(f^m(y)_{\underline{(\lambda,\mu)}}\right) = \sum_{s+t=n-1} \left((y)_{\underline{\lambda}_{nn-1\dots n-s}}\right)_{\underline{\mu}_{nn-1\dots n-t}}, \tag{66}$$

where $\underline{\lambda}_{nn-1\cdots n-s} = (\lambda_0,\dots,\lambda_{n-1})_{n-1\cdots n-s} = \cdots = (\lambda_0,\dots,\lambda_{n-s-1})$ is the successive deletion of the terms $\lambda_n,\dots,\lambda_{n-s}$, and

$$\left(g^m(x)_{\underline{\lambda}}\right)_{\underline{\mu}} = \sum_{(l,k)} sign(l,k) x_{\underline{(\lambda^{l_1\dots l_t},\mu^{k_1\dots k_s})}}, \tag{67}$$

where $\underline{\lambda} \in \Lambda^s$, $\underline{\mu} \in M^t$, $\underline{\lambda}^{l_1\dots l_t} = (\lambda_0,\dots,\lambda_{l_1},\lambda_{l_1},\dots,\lambda_{l_t}\lambda_{l_t},\dots,\lambda_s)$ is the successive t-fold application of the degeneracy operator on the places $(l_1,\dots,l_t)$ and $\underline{\mu}^{k_1\cdots k_s} = (\mu_0,\dots,\mu_{k_1},\mu_{k_1},\dots,\mu_{k_s}\mu_{k_s},\dots,\mu_t)$ is the successive s-fold application of the degeneracy operator to the different places $(k_1,\dots,k_s)$ and the summation of all (s,t)-shuffles (l,k). By the degeneracy operator we mean $\underline{\lambda}^j$ obtained from $\underline{\lambda}$ by repeating λ_j.

We omit the tedious verification that f^m and g^m are cochain mappings and refer to similar situation in [3]. Notice only that, for $n = 0$, f^m and g^m are equalities on $C^m_{(\lambda_0,\mu_0)}$

therefore, by the acyclic model principle, $f^m g^m$ and $g^m f^m$ are cochain homotopic to $1_{(\mathrm{ho}\varprojlim_{(\lambda,\mu)}\underline{C})^m}$ and $1_{(\mathrm{ho}\varprojlim_\lambda \mathrm{ho}\varprojlim_\mu \underline{C})^m}$, respectively.

Remark 13. Let s, t be nonnegative integer. By the (s,t)-shuffle (l,k) we understand two disjoint subsets $1 \leqslant l_1 < l_2 < \cdots < l_s \leqslant s+t$, $1 \leqslant k_1 < k_2 < \cdots < k_t \leqslant s+t$ of the set $\{1,\ldots,s+t\}$. By *sign*(l,k) we mean the sign of the permutation $(l_1,\ldots,l_s,k_1,\ldots,k_t)$ of the numbers $\{1,\ldots,s+t\}$.

Theorem 6. *Let $\underline{A} = (A_{(\lambda,\mu)}, p_{(\lambda,\mu)(\lambda',\mu')}, \Lambda \times M)$ be an inverse system of Abelian groups and homomorphisms, indexed by a product $\Lambda \times M$ of cofinite partially ordered directed sets Λ and M. Then there exist two spectral sequence $E_2^{s,t} = \varprojlim_\lambda^s \varprojlim_\mu^t A_{(\lambda,\mu)}$ and ${}'E_2^{s,t} = \varprojlim_\mu^s \varprojlim_\lambda^t A_{(\lambda,\mu)}$ strongly converging to $\varprojlim_{(\lambda,\mu)}^{s+t} A_{(\lambda,\mu)}$.*

Proof. We shall describe only the first one $E^{s,t}$, clearly, the second one ${}'E^{s,t}$ is symmetric. Consider the cochain complex $\underline{C}_\lambda = \mathrm{ho}\varprojlim_\mu \underline{A}$. By Proposition 6, $H^t(\underline{C}_\lambda) = \varprojlim_\mu^t \underline{A}$. Since, by Proposition 3, holim is a covariant functor, $(\underline{C}_\lambda, p^\#_{\lambda\lambda'}, \Lambda)$ is an inverse system of cochain complexes. We then consider $\mathrm{ho}\varprojlim_\lambda(\underline{C}_\lambda, p^\#_{\lambda\lambda'}, \Lambda)$, the bicomplex

$$D^{s,t} = \prod_{\underline{\lambda}\in\Lambda^s} (\underline{C}_{\lambda_0})^t = \prod_{\underline{\lambda}\in\Lambda^s}\prod_{\underline{\mu}\in M^t} A_{(\lambda_0,\mu_0)}$$

and the spectral sequence $E_r^{s,t}$ with $E_2^{s,t} = \varprojlim_\lambda^s H^t(\underline{C}_\lambda) = \varprojlim_\lambda^s \varprojlim_\mu^t \underline{A}$, which conditionally converges to

$$H^{s+t}\big(\mathrm{ho}\varprojlim_\lambda \underline{C}_\lambda\big) = H^{s+t}\big(\mathrm{ho}\varprojlim_\lambda \mathrm{ho}\varprojlim_\mu \underline{A}\big).$$

Since for $s < 0$ and $t < 0$, $D^{s,t} = 0$, the $\prod$-total bicomplex becomes $\sum$-total and $E^{s,t}$ is a first quadrant spectral sequence, which as it is well known, strongly converges to the cohomology group of the $\sum$-total complex, i.e., to $H^{s+t}(\mathrm{ho}\varprojlim_\lambda \mathrm{ho}\varprojlim_\mu \underline{A})$. By Theorem 5, $\mathrm{ho}\varprojlim_\lambda \mathrm{ho}\varprojlim_\mu \underline{A}$ is weak equivalent to $\mathrm{ho}\varprojlim_{(\lambda,\mu)} \underline{A}$ and thus, this spectral sequence strongly converges to

$$H^{s+t}\big(\mathrm{ho}\varprojlim_{(\lambda,\mu)} \underline{A}\big) = \mathrm{ho}\varprojlim_{(\lambda,\mu)}^{s+t} \underline{A}. \qquad \square$$

Remark 14. It is well known (see, e.g., [2]) that, if some strongly converging spectral sequence has only two non-trivial vertical lines, e.g., $E_r^{u,v} = 0$, for all $v \neq q, q'$, where $r \geqslant 2$ and $q' - q \geqslant r - 1$, then one has the following long exact sequence:

$$\begin{aligned} \cdots \to\ & E_r^{m-q,q} \to H^m \to E_r^{m-q',q'} \to\ E_r^{m+1-q,q} \\ \to\ & H^{m+1} \to E_r^{m+1-q',q'} \to \cdots, \end{aligned} \tag{68}$$

which in the case of interest to us (see below), i.e., when $r = 2$, $q = 0$ and $q' = 1$, becomes the following exact sequence:

$$\begin{aligned}
0 \to\ & \varprojlim_\lambda \varprojlim_\mu A_{(\lambda,\mu)} \to \varprojlim_{(\lambda,\mu)} A_{(\lambda,\mu)} \to 0 \\
\to\ & \varprojlim{}^1_\lambda \varprojlim_\mu A_{(\lambda,\mu)} \to \varprojlim{}^1_{(\lambda,\mu)} A_{(\lambda,\mu)} \to \varprojlim_\lambda \varprojlim{}^1_\mu A_{(\lambda,\mu)} \\
\to\ & \varprojlim{}^2_\lambda \varprojlim_\mu A_{(\lambda,\mu)} \to \varprojlim{}^2_{(\lambda,\mu)} A_{(\lambda,\mu)} \to \varprojlim{}^1_\lambda \varprojlim{}^1_\mu A_{(\lambda,\mu)} \\
\to\ & \varprojlim{}^3_\lambda \varprojlim_\mu A_{(\lambda,\mu)} \to \varprojlim{}^3_{(\lambda,\mu)} A_{(\lambda,\mu)} \to \varprojlim{}^2_\lambda \varprojlim{}^1_\mu A_{(\lambda,\mu)} \to \cdots.
\end{aligned} \tag{69}$$

If the index set $\Lambda = N$ is countable, then (69) becomes a short exact sequence:

$$\begin{aligned}
0 \to\ & \varprojlim_n \varprojlim_\mu A_{(n,\mu)} \to \varprojlim_{(n,\mu)} A_{(n,\mu)} \to 0 \\
\to\ & \varprojlim{}^1_n \varprojlim_\mu A_{(n,\mu)} \to \varprojlim{}^1_{(n,\mu)} A_{(n,\mu)} \to \varprojlim_n \varprojlim{}^1_\mu A_{(n,\mu)} \to 0.
\end{aligned} \tag{70}$$

6. Proof of the Second Structure Theorem

We have already prepared everything for proving Theorem 3. By Theorem 4, there exists a homotopy inverse system $\underline{Y} = (Y_{(\lambda,\mu)}, q_{(\lambda,\mu)(\lambda',\mu')}, \Lambda \times M)$ of compact polyhedra $Y_{(\lambda,\mu)}$ and bonding maps $q_{(\lambda,\mu)(\lambda',\mu')}$, indexed by a product $\Lambda \times M$ of cofinite partially ordered directed sets Λ and M such that for every $\lambda \in \Lambda$, $\underline{q}_\lambda : X_\lambda \to \underline{Y}_\lambda$ is an inverse limit and $\underline{p} = \underline{q}_\lambda p_\lambda : X \to \underline{Y}$ is an ANR-expansion in the sense of Morita.

Now consider the corresponding inverse system $(H_m(Y_{(\lambda,\mu)}; G), p_{(\lambda,\mu)(\lambda',\mu')*}, \Lambda \times M)$ of homology groups and homomorphisms of finite polyhedra $Y_{(\lambda,\mu)}$, $m \in \mathbb{Z}$. For each fixed $m \geqslant 0$, by Theorem 6, there exists a spectral sequence $E_2^{s,t} = \varprojlim{}^s_\lambda \varprojlim{}^t_\mu H_m(Y_{(\lambda,\mu)}; G)$ strongly converging to $\varprojlim{}^{s+t}_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)}; G)$. Since by Theorem 2, $\varprojlim{}^t_\mu H_m(Y_{(\lambda,\mu)}; G) = 0$ for all $t \geqslant 2$, $E_2^{s,t} \neq 0$ only for $t = 0, 1$ and thus, by Remark 14 and (69), we obtain the following long exact sequence

$$\begin{aligned}
0 \to\ & \varprojlim_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)}; G) \to \varprojlim_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)}; G) \to 0 \\
\to\ & \varprojlim{}^1_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)}; G) \to \varprojlim{}^1_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)}; G) \\
\to\ & \varprojlim_\lambda \varprojlim{}^1_\mu H_m(Y_{(\lambda,\mu)}; G) \to \varprojlim{}^2_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)}; G) \\
\to\ & \varprojlim{}^2_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)}; G) \to \varprojlim{}^1_\lambda \varprojlim{}^1_\mu H_m(Y_{(\lambda,\mu)}; G) \\
\to\ & \varprojlim{}^3_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)}; G) \to \varprojlim{}^3_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)}; G) \\
\to\ & \varprojlim{}^2_\lambda \varprojlim{}^1_\mu H_m(Y_{(\lambda,\mu)}; G) \to \cdots.
\end{aligned} \tag{71}$$

Again by Theorem 2, $\varprojlim^s_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)};G) = 0$, for all $s \geqslant 2$, and by the assumption of Theorem 3, i.e., $\varprojlim^s_\lambda \check{H}_m(X_\lambda;G) = 0$, for every $s \geqslant 2$, taking into account that

$$\varprojlim_\mu H_m(Y_{(\lambda,\mu)};G) = \check{H}_m(X_\lambda;G)$$

and

$$\varprojlim_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)};G) = \check{H}_m(X;G),$$

(71) turns into the following short exact sequence

$$\begin{aligned} 0 \;\to\; & \varprojlim^1_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)};G) \to \varprojlim^1_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)};G) \\ \to\; & \varprojlim_\lambda \varprojlim^1_\mu H_m(Y_{(\lambda,\mu)};G) \to 0, \end{aligned} \tag{72}$$

which is the first column in (18), the isomorphism

$$\varprojlim_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)};G) \approx \varprojlim_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)};G), \tag{73}$$

which is the third column in (18), and the equalities

$$\varprojlim^s_\lambda \varprojlim^1_\mu H_m(Y_{(\lambda,\mu)};G) = 0, \tag{74}$$

for all $s \geqslant 1$.

Now we apply the $\varprojlim_\lambda$-functor to the Milnor sequence

$$0 \to \varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)};G) \to \overline{H}_m(X_\lambda;G) \to \varprojlim_\mu H_m(Y_{(\lambda,\mu)},G) \to 0 \tag{75}$$

and obtain the following long exact sequence

$$\begin{aligned} 0 \;\to\; & \varprojlim_\lambda \varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)};G) \to \varprojlim_\lambda \overline{H}_m(X_\lambda;G) \\ \to\; & \varprojlim_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)};G) \to \varprojlim^1_\lambda \varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)};G) \\ \to\; & \varprojlim^1_\lambda \overline{H}_m(X_\lambda;G) \leftarrow \varprojlim^1_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)};G) \\ \to\; & \varprojlim^2_\lambda \varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)};G) \to \varprojlim^2_\lambda \overline{H}_m(X_\lambda;G) \\ \to\; & \varprojlim^2_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)};G) \leftarrow \cdots. \end{aligned} \tag{76}$$

Now, by (74) and the assumption of Theorem 3, i.e., $\varprojlim^s_\lambda \check{H}_m(X_\lambda;G) = 0$, for every $s \geqslant 2$, (76) turns into the following short exact sequence

$$\begin{aligned} 0 \;\to\; & \varprojlim_\lambda \varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)};G) \\ \to\; & \varprojlim_\lambda \overline{H}_m(X_\lambda;G) \to \varprojlim_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)};G) \to 0, \end{aligned} \tag{77}$$

which is the third row in (18), the isomorphism

$$\varprojlim^1_\lambda \overline{H}_m(X_\lambda;G) \approx \varprojlim^1_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)};G), \tag{78}$$

which is the first row in (18) and the equalities

$$\varprojlim\nolimits_{\lambda}^{s} \overline{H}_m(X_\lambda; G) = 0, \tag{79}$$

for all $s \geqslant 2$.

By the result of Mdzinarishvili (see [17]), presented without any condition for inverse limits of compact Hausdorff spaces $X = \varprojlim_\lambda X_\lambda$ the following long exact sequence

$$\begin{aligned} \cdots &\to \varprojlim\nolimits_{\lambda}^{3} \overline{H}_{m+2}(X_\lambda; G) \to \varprojlim\nolimits_{\lambda}^{1} \overline{H}_{m+1}(X_\lambda; G) \to \overline{H}_m(X; G) \\ &\to \varprojlim\nolimits_{\lambda} \overline{H}_m(X_\lambda; G) \to \varprojlim\nolimits_{\lambda}^{2} \overline{H}_{m+1}(X_\lambda; G) \to \cdots \end{aligned} \tag{80}$$

for Milnor homology $\overline{H}_*$ (see Definition in [19]) and thus, for strong homology, because they coincides (see [24, p. 110], and also [18, p. 168]).

Then by (79), formula (80) turns into the following short exact sequence

$$0 \to \varprojlim\nolimits_{\lambda}^{1} \overline{H}_{m+1}(X_\lambda; G) \to \overline{H}_m(X; G) \to \varprojlim\nolimits_{\lambda} \overline{H}_m(X_\lambda; G) \to 0, \tag{81}$$

which is the second column in (18).

Choose an ANR-resolution $\underline{p}' : X \to \underline{Y}' = (Y'_\nu, r_{\nu\nu'}, N)$ of X in the sense of Mardešić which consists of compact polyhedra Y_ν and write the following Milnor short exact sequence (actually up to notation it is the second column in (7))

$$0 \to \varprojlim\nolimits_{\nu}^{1} H_{m+1}\big(Y'_\nu; G\big) \to \overline{H}_m(X; G) \to \varprojlim\nolimits_{\nu} H_m\big(Y'_\nu; G\big) \to 0. \tag{82}$$

On the other hand, by Proposition 1, the identity mapping $i : X \to X$ induces an isomorphism $\underline{i} : \underline{Y}' \to \underline{Y}$ in *pro-HTop* and hence, $\underline{i}$ induces isomorphisms

$$\underline{i}_* : \big(H_m\big(Y'_\nu; G\big), r_{\nu\nu'*}, N\big) \to \big(H_m(Y_{\lambda,\mu}), q_{(\lambda,\mu)(\lambda,\lambda')*}, \Lambda \times M\big)$$

of pro-groups, for all m, and thus, by corollary to Proposition 6, the following isomorphisms

$$\varprojlim\nolimits_{\nu} H_m(Y'_\nu; G) \approx \varprojlim\nolimits_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)}; G) \tag{83}$$

and

$$\varprojlim\nolimits_{\nu}^{1} H_{m+1}\big(Y'_\nu; G\big) \approx \varprojlim\nolimits_{(\lambda,\mu)}^{1} H_{m+1}(Y_{(\lambda,\mu)}; G). \tag{84}$$

Thus, by (83) and (84), formula (82) turns into the following exact sequence

$$0 \to \varprojlim\nolimits_{(\lambda,\mu)}^{1} H_{m+1}(Y_{(\lambda\mu)}; G) \to \overline{H}_m(X; G) \to \varprojlim\nolimits_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)}; G) \to 0, \tag{85}$$

which is the second row in (18), because by definition

$$\check{H}_m(X; G) = \varprojlim\nolimits_{(\lambda,\mu)} H_m(Y_{(\lambda,\mu)}; G)$$

since $\underline{p} : X \to \underline{Y}$ is an ANR-expansion of X in the sense of Morita.

Now, for each $m \geqslant 0$, we also need the following two well-known Roos type spectral sequences (see, e.g., [14]) for the direct system $\underline{A} = (\check{H}^m(X_\lambda), p_{\lambda\lambda'\#}, \Lambda)$ strongly converging to $Ext^n(\varinjlim_\lambda \check{H}^m(X_\lambda), G)$ and to $Pext^n(\varinjlim_\lambda \check{H}^m(X_\lambda), G)$, respectively. Since for arbitrary Abelian groups A and G, $Ext^n(A, G) = 0$ and $Pext^n(A, G) = 0$, for $n \geqslant 2$, these two spectral sequences turn into the following isomorphisms and exact sequences

$$\varprojlim_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \approx Hom\big(\check{H}^m(X), G\big), \tag{86}$$

$$\varprojlim{}^s_\lambda Ext\big(\check{H}^m(X_\lambda), G\big) \approx \varprojlim{}^{s+2}_\lambda Hom\big(\check{H}^m(X_\lambda), G\big), \quad s \geqslant 1, \tag{87}$$

$$\begin{aligned} 0 &\to \varprojlim{}^1_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \to Ext\big(\check{H}^m(X), G\big) \\ &\to \varprojlim_\lambda Ext\big(\check{H}^m(X_\lambda), G\big) \to \varprojlim{}^2_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \to 0. \end{aligned} \tag{88}$$

$$\varprojlim{}^s_\lambda Pext\big(\check{H}^m(X_\lambda), G\big) \approx \varprojlim{}^{s+2}_\lambda Hom\big(\check{H}^m(X_\lambda), G\big), \quad s \geqslant 1, \tag{89}$$

$$\begin{aligned} 0 &\to \varprojlim{}^1_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \to Pext\big(\check{H}^m(X), G\big) \\ &\to \varprojlim_\lambda Pext\big(\check{H}^m(X_\lambda), G\big) \to \varprojlim{}^2_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \to 0. \end{aligned} \tag{90}$$

Remark 15. The second spectral sequence concerning *Pext* belongs to Kuz'minov [5] and as we see below is dual to ours in Theorem 6, because it describes the same target group, just as the dual canonical sequences (46) and (47) describe the cohomology group $H^m(C)$ or the dual injective and projective unraveled exact couples (58) and (59) give the same spectral sequence (60) and describe the target group $H^m(\mathrm{ho}\varprojlim_\lambda \underline{C})$ identically, but in a dual form (see [7]). Nevertheless, the spectral sequence in Theorem 6 has an advantage over *Pext*-exact sequence, because it can be used for arbitrary inverse systems of homotopy groups, although the analogues to the *Pext* construction for cohomotopies has not yet been found.

Taking into account (see (1)) that

$$Pext\big(\check{H}^m(X_\lambda), G\big) \approx \varprojlim{}^1_\mu H_m\big(Y_{(\lambda,\mu)}; G\big)$$

and

$$Pext\big(\check{H}^m(X), G\big) \approx \varprojlim{}^1_{(\lambda,\mu)} H_m\big((Y_{(\lambda,\mu)}; G\big), \quad \text{for each } \lambda \in \Lambda \text{ and } m > 0,$$

we see that (72), (78), (87) and (88) imply $\varprojlim{}^2_\lambda Hom(\check{H}^m(X_\lambda), G) = 0$ and thus imply the following short exact sequence

$$\begin{aligned} 0 &\to \varprojlim{}^1_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \to Ext\big(\check{H}^m(X), G\big) \\ &\to \varprojlim_\lambda Ext\big(\check{H}^m(X_\lambda), G\big) \to 0, \end{aligned} \tag{91}$$

which coincide, for $m+1$, with the first column in (19), the following short exact sequence

$$\begin{aligned} 0 &\to \varprojlim{}^1_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \to Pext\big(\check{H}^m(X), G\big) \\ &\to \varprojlim_\lambda Pext\big(\check{H}^m(X_\lambda), G\big) \to 0, \end{aligned} \tag{92}$$

which coincides (more precisely, it is isomorphic and that is what we mean in Remark 20 by a dual form) with the first column in (18) and the equalities

$$\varprojlim{}^s_\lambda Ext\big(\check{H}^m(X_\lambda), G\big) = 0, \tag{93}$$

for $s \geqslant 1$ and

$$\varprojlim{}^s_\lambda Pext\big(\check{H}^m(X_\lambda), G\big) = 0 \tag{94}$$

for $s \geqslant 1$, which coincides with (74).

We now apply the functor $\varprojlim_\lambda$ to the universal coefficient formula

$$0 \to Ext\big(\check{H}^{m+1}(X_\lambda), G\big) \to \overline{H}_m(X_\lambda; G) \to Hom\big(\check{H}^m(X_\lambda), G\big) \to 0 \tag{95}$$

and obtain the following long exact sequence

$$\begin{aligned} 0 \;\to\; & \varprojlim{}_\lambda Ext\big(\check{H}^{m+1}(X_\lambda), G\big) \to \varprojlim{}_\lambda \overline{H}_m(X_\lambda; G) \\ \to\; & \varprojlim{}_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \to \varprojlim{}^1_\lambda Ext\big(\check{H}^{m+1}(X_\lambda), G\big) \\ \to\; & \varprojlim{}^1_\lambda \overline{H}_m(X_\lambda; G) \to \varprojlim{}^1_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \\ \to\; & \varprojlim{}^2_\lambda Ext\big(\check{H}^{m+1}(X_\lambda), G\big) \to \cdots. \end{aligned} \tag{96}$$

By (93), it turns into the following short exact sequence

$$\begin{aligned} 0 \;\to\; & \varprojlim{}_\lambda Ext\big(\check{H}^{m+1}(X_\lambda), G\big) \to \varprojlim{}_\lambda \overline{H}_m(X_\lambda; G) \\ \to\; & \varprojlim{}_\lambda Hom(\check{H}^m(X_\lambda), G) \to 0, \end{aligned} \tag{97}$$

which is the third row in (19), and in the isomorphism

$$\varprojlim{}^1_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \approx \varprojlim{}^1_\lambda \overline{H}_m(X_\lambda; G), \tag{98}$$

which, for $m+1$, is the first row of (19).

The second row in (19) is nothing else but the universal coefficient formula and the third column in (19) follows from the commutativity of the functor *Hom* with the functors $\varprojlim$ and $\varinjlim$ and the continuity of Čech cohomology, i.e.,

$$\varprojlim{}_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \approx Hom\big(\varinjlim{}_\lambda \check{H}^m(X_\lambda), G\big) \approx Hom\big(\check{H}^m(X), G\big).$$

Diagram (20) almost coincides with (7) when we notice that the pro-group $(H_{m+1}(X_\lambda; G))$ in (7) is isomorphic in the pro-category *pro-Ab* to the pro-group $(H_{m+1}(Y_{(\lambda,\mu)}; G))$ in (20), since $\underline{p}: X \to \underline{Y}$ is an ANR-expansion in the sense of Morita, by corollary to Proposition 4,

$$\varprojlim{}^1_{(\lambda,\mu)} H_{m+1}(Y_{(\lambda,\mu)}; G) \approx \varprojlim{}^1_\lambda H_{m+1}(X_\lambda; G).$$

The commutativity of (20) was proved in [14]. In the same way one can check the commutativity of diagrams (18) and (19) taking into account the naturality of all homomorphisms in all of our formulae.

Since $\underline{q}_{\lambda} : X_\lambda \to \underline{Y}_\lambda$, $\lambda \in \Lambda$, are compact polyhedral resolutions of X_λ, we consider diagram (7) by putting $Y_{(\lambda,\mu)}$ on the place of X_λ. Then, applying the functor $\varprojlim_\lambda$ to the first exact column

$$0 \to \mathit{Pext}\big(\check{H}^{m+1}(X_\lambda), G\big) \to \mathit{Ext}\big(\check{H}^{m+1}(X_\lambda), G\big)$$
$$\to \varprojlim_\mu \mathit{Ext}\big(\check{H}^{m+1}(Y_{(\lambda,\mu)}), G\big) \to 0 \tag{99}$$

we obtain the following exact sequence

$$0 \to \varprojlim \mathit{Pext}\big(\check{H}^{m+1}(X_\lambda), G\big) \to \varprojlim_\lambda \mathit{Ext}\big(\check{H}^{m+1}(X_\lambda), G\big)$$
$$\to \varprojlim_\lambda \varprojlim_\mu \mathit{Ext}\big(\check{H}^{m+1}(Y_{(\lambda,\mu)}), G\big) \to 0, \tag{100}$$

which is the first column in (21), and we also obtain the following equalities

$$\varprojlim{}^s_\lambda \varprojlim_\mu \mathit{Ext}\big(\check{H}^{m+1}(Y_{(\lambda,\mu)}), G\big) = 0, \tag{101}$$

for all $s \geqslant 1$.

Now for the third row in (7), i.e., for the following short exact sequence

$$0 \to \varprojlim_\mu \mathit{Ext}\big(\check{H}^{m+1}(Y_{(\lambda,\mu)}), G\big) \to \check{H}_m(X_\lambda, ; G)$$
$$\to \varprojlim_\mu \mathit{Hom}\big(\check{H}^m(Y_{(\lambda,\mu)}), G\big) \to 0, \tag{102}$$

we apply the functor $\varprojlim_\lambda$ and, taking into account (101), we obtain the following exact sequence

$$0 \to \varprojlim_\lambda \varprojlim_\mu \mathit{Ext}\big(\check{H}^{m+1}(Y_{(\lambda,\mu)}), G\big) \to \varprojlim_\lambda \check{H}_m(X_\lambda, ; G)$$
$$\to \varprojlim_\lambda \varprojlim_\mu \mathit{Hom}\big(\check{H}^m(Y_{(\lambda,\mu)}), G\big) \to 0, \tag{103}$$

which is the third row in (21) because of commutativity of Čech homology, i.e., because of $\varprojlim_\lambda \check{H}_m(X_\lambda; G) \approx \check{H}_m(X; G)$.

Now for the second row in (7), i.e., for the following short exact sequence

$$0 \to \varprojlim_\mu \mathit{Ext}\big(\check{H}^{m+1}(X_\lambda), G\big) \to \varprojlim_\mu \overline{H}_m(X_\lambda; G)$$
$$\to \mathit{Hom}\big(\check{H}^m(X_\lambda), G\big) \to 0, \tag{104}$$

we apply the functor $\varprojlim_\lambda$ and, taking into account (101), we obtain the following exact sequence

$$0 \to \varprojlim_\lambda \mathit{Ext}\big(\check{H}^{m+1}(X_\lambda), G\big) \to \varprojlim_\lambda \overline{H}_m(X_\lambda; G)$$
$$\to \varprojlim_\lambda \mathit{Hom}\big(\check{H}^m(X_\lambda), G\big) \to 0, \tag{105}$$

which is the second row in (21), because of the commutativity of the functors $\varprojlim$ and *Hom* and $\varinjlim$ and of the continuity of Čech cohomology, i.e.,

$$\varprojlim_\lambda \mathit{Hom}\big(\check{H}^m(X_\lambda), G\big) \approx \mathit{Hom}\big(\varinjlim_\lambda \check{H}^m(X_\lambda), G\big) \approx \mathit{Hom}\big(\check{H}^m(X; G)\big).$$

Now for the second column in (7), i.e., for the following short exact sequence

$$0 \to \varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)}; G) \to \varprojlim_\mu \overline{H}_m(X_\lambda; G) \to \check{H}_m(X_\lambda; G) \to 0, \tag{106}$$

we apply the functor $\varprojlim_\lambda$ and, taking into account (74), we obtain the following exact sequence

$$\begin{aligned} 0 \to\ & \varprojlim_\lambda \varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)}; G) \to \varprojlim_\lambda \overline{H}_m(X_\lambda; G) \\ \to\ & \varprojlim_\lambda \varprojlim_\mu \check{H}_m(X_\lambda; G) \to 0, \end{aligned} \tag{107}$$

which is the second column in (21), because of the continuity of Čech homology, i.e., $\varprojlim_\lambda \check{H}_m(X_\lambda; G) \approx \check{H}_m(X; G)$.

The first row in (21) is clear, since by (1),

$$Pext\big(\check{H}^{m+1}(X_\lambda), G\big) \approx \varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)}; G). \tag{108}$$

The third column in (21) is also clear, since as we have already seen,

$$\begin{aligned} & \varprojlim_\lambda \varprojlim_\mu Hom\big(H^m(Y_{(\lambda,\mu)}), G\big) \approx \varprojlim_\lambda Hom\big(\varinjlim_\mu H^m(Y_{(\lambda,\mu)}), G\big) \\ & \quad \approx \varprojlim_\lambda Hom\big(\check{H}^m(X_\lambda), G\big) \approx Hom\big(\varinjlim_\lambda H^m(X_\lambda), G\big) \\ & \quad \approx Hom\big(\check{H}^m(X), G\big). \end{aligned}$$

At last we put $F_1 = Ker(\gamma)$, $F_2 = Ker(\beta)$ and $F_3 = Ker(\alpha)$, where α, β, γ in (18)–(20). This completes the proof of Theorem 3. □

Remark 16. Note that the assumptions in Theorem 3 are equivalent to $\varprojlim^s_\lambda Hom(\check{H}^m(X_\lambda), G) = 0$, for all $s \geqslant 2$ and integers $m \geqslant 0$.

Proof of Corollary 1. It is well known that for a countable index set $\Lambda = N$, $\varprojlim^s_n \check{H}_m(X_\lambda; G) = 0$ for all $s \geqslant 2$ and each integer m. Now we use Theorem 3. □

Proof of Corollary 2. It is well known that if the groups $\check{H}_m(X_\lambda; G)$ are finitely generated, then $\varprojlim^s_\lambda \check{H}_m(X_\lambda; G) = 0$, for all $s \geqslant 2$ and each integer m, and we apply Theorem 3. □

Proof of Corollary 3. If $\overline{H}_m(X_\lambda; G) = \check{H}_m(X_\lambda; G)$ for all $m \geqslant 0$, then by the Milnor formula (75), taking into account that $\varprojlim^1_\mu H_0(Y_{(\lambda,\mu)}; G) = 0$, we obtain the following equality

$$\varprojlim^1_\mu H_{m+1}(Y_{(\lambda,\mu)}; G) = 0, \tag{109}$$

for all $m \geqslant 0$. Then by (71) and (107) we obtain

$$\varprojlim^s_\lambda \check{H}_m(X_\lambda; G) = \varprojlim^s_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)}; G) = 0, \quad \text{for all } s \geqslant 2.$$

Now apply again Theorem 3. □

Remark 17. Corollary 3 is a generalization of the result in [25, p. 131] where the case of a compact Hausdorff HLC-neighborhoods in I^τ of compact X were considered.

Prrof of Corollary 4. It is well known that, if the groups $\check{H}^m(X_\lambda)$ are finitely generated, then $\varprojlim^s_\lambda Hom(\check{H}^m(X_\lambda), G) = 0$, for all $s \geqslant 2$ and each integer m. Then, by (71) and (88), taking into account that $\varprojlim^1_\mu H_0(Y_{(\lambda,\mu)}; G) = 0$, we obtain

$$\varprojlim^s_\lambda \check{H}_m(X_\lambda; G) = \varprojlim^s_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)}; G) = 0, \quad \text{for all } s \geqslant 2$$

and again we apply Theorem 3. □

Proof of Corollary 5. If $\overline{H}_m(X_\lambda; G) = Hom(\check{H}^m(X_\lambda), G)$, then it follows from the universal coefficients formula (95) that $Ext(\check{H}_{m+1}(X_\lambda), G) = 0$, for all $m \geqslant 0$. Therefore, by (87), (88) and (71), taking into account that $\varprojlim^1_\mu H_0(Y_{(\lambda,\mu)}; G) = 0$, we obtain

$$\varprojlim^s_\lambda \check{H}_m(X_\lambda; G) = \varprojlim^s_\lambda \varprojlim_\mu H_m(Y_{(\lambda,\mu)}; G) = 0,$$

for all $s \geqslant 2$ and we apply again Theorem 3. □

7. Conclusion

By (108) and the definition of Čech homology, diagram (18) can be rewritten as the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \varprojlim^1_\lambda \check{H}_{m+1}(X_\lambda; G) \arrow[r, "\approx"] \arrow[d] & \varprojlim^1_\lambda \overline{H}_{m+1}(X_\lambda; G) \arrow[rr] \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & Pext(\check{H}^{m+1}(X), G) \arrow[r] \arrow[d] & \overline{H}_m(X; G) \arrow[rr, "\beta"] \arrow[d, "\gamma"] & & \check{H}_m(X; G) \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & \varprojlim_\lambda Pext(\check{H}^{m+1}(X_\lambda), G) \arrow[r] \arrow[d] & \varprojlim_\lambda \overline{H}_m(X_\lambda; G) \arrow[rr] \arrow[d] & & \varprojlim_\lambda \check{H}_m(X_\lambda, G) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 &
\end{tikzcd} \tag{110}$$

which together with diagram (19) make a 3-dimensional diagram, whose intersection is the following short exact sequence

$$0 \to \varprojlim^1_\lambda \overline{H}_{m+1}(X_\lambda; G) \to \overline{H}_m(X; G) \to \varprojlim_\lambda \overline{H}_m(X_\lambda; G) \to 0. \tag{111}$$

The formula (111) as compared with the Milnor formula is variable, i.e., it depends on the choice of the inverse system $\underline{X} = (X_\lambda, p_{\lambda,\lambda'}, \Lambda)$. For example, if all $X_\lambda = X$, then

$F_1 = 0$ and if all X_λ are ANR-spaces or have the homotopy type of compact polyhedra, then $F_1 = F_2$. Therefore, we raise the following

Problem 1. Let F be an arbitrary subgroup in $\overline{H}_m(X; G)$ such that $0 \subset F \subset F_2$. Does there exist an inverse system $\underline{X} = (X_\lambda, p_{\lambda,\lambda'}, \Lambda)$ such that $X = \varprojlim_\lambda \underline{X}$ and $F_1 = F$?

If G is finitely generated, then for every space X there exists another universal coefficient formula

$$0 \to \overline{H}_m(X; \mathbb{Z}) \otimes G \to \overline{H}_m(X; G) \to \mathit{Tor}\big(\overline{H}_{m-1}(X; \mathbb{Z}), G\big) \to 0. \tag{112}$$

Indeed, it can be proved separately for $\mathbb{Z}$ and $\mathbb{Z}/n$, while any finitely generated G is a direct sum of such groups. For compact Hausdorff space X the proof see, e.g., in [23].

Now let $X = \varprojlim_\lambda \underline{X}$, where X_λ, $\lambda \in \Lambda$ are compact Hausdorff spaces, and each such space X_λ we consider the universal coefficient formula (112).

Problem 2. What is the relation between all these groups in (112) for X and X_λ?

Problem 3. Is the Second Structure Theorem valid for arbitrary partially ordered directed index set Λ?

Remark 18. The Second Structure Theorem is not valid without any assumption on homology of X_λ, because there is an example (see [26, p. 27]) of an inverse system of compact Hausdorff spaces X_λ such that (111) is not exact.

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